Math 105 — Second Midterm

March 15, 2013

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 9 pages including this cover AND IS DOUBLE SIDED. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions.
5. Show an appropriate amount of work (including appropriate explanation). Include units in your answer where that is appropriate. Time is of course a consideration, but do not provide no work except when specified.
6. You may use any previously permitted calculator. However, you must state when you use it.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones and hats.
9. Remember that this is a chance to show what you've learned, and that the questions are just prompts.

Problem	Points	Score
1	15	
2	18	
3	10	
4	13	
5	14	
6	14	
7	14	
8	02	
Total	100	

1. [15 points] Suppose that $W(h)$ is an invertible function which tells us how many gallons of water an oak tree of height h feet uses on a hot summer day.

a. [6 points] Give a practical (one sentence) interpretation for each of the following quantities or statements.

- $W(50) = 25$.
- $W^{-1}(40)$
- $\frac{d}{dx}[W^{-1}](40) = 2$

Solution:

- A tree of height 50 feet uses 25 gallons on a hot summer day.
- This gives the height of a tree which uses 40 gallons of water on a hot summer day.
- A tree which uses 41 gallons of water on a hot summer day is approximately 2 feet taller than a tree which uses 40 gallons of water.

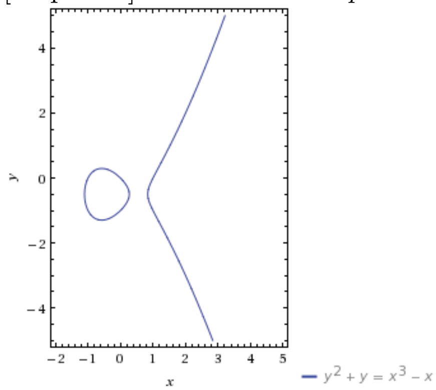
b. [9 points] Use the following table to calculate $\frac{d}{dx}[W^{-1}](40)$, and $W^{-1}(30)$ and $W'(50)$.

h	10	20	30	40	50	60
$W(h)$	5	15	30	35	40	70
$W'(h)$	2	1	2	4	1	5

Solution:

- $W'(50)$ can be read off the chart as 1.
- $W^{-1}(30)$ is the number which gives 30 as the output of W so it is 30.
- $W^{-1'}(40) = \dfrac{1}{W'(W^{-1}(40))} = \dfrac{1}{W'(50)} = 1.$

2. [18 points] There is an *elliptic curve* given by the equation $y^2 + y = x^3 - x$.



a. [2 points] Find any point that is NOT on the curve. Prove that it is not on the curve.

Solution: $(1, 1)$ is not on the curve because when we plug in 1 for x and y we get $1^2 + 1 = 1^3 - 1$ which is not true.

b. [2 points] Check that the point $(2, 2)$ is on the given elliptic curve.

Solution: When we plug in 2 for both x and y we get $2^2 + 2 = 2^3 - 2$ which are both 6 so it is correct.

c. [8 points] Find the slope of the tangent line at $(2, 2)$.

Solution: Pretending that $y = f(x)$ we get

$$(f(x))^2 + f(x) = x^3 - x$$

then we differentiate both sides

$$2f(x)f'(x) + f'(x) = 3x^2 - 1$$

switching back to y gives

$$\begin{aligned} 2y\frac{dy}{dx} + \frac{dy}{dx} &= 3x^2 - 1 \\ \frac{dy}{dx} &= \frac{3x^2 - 1}{2y + 1}. \end{aligned}$$

If we now plug in 2 for x and y we get the slope is $\frac{11}{5}$.

d. [6 points] Use the previous parts to approximate the value of x on the curve when $y = 2.5$.

Solution: We find the tangent line by setting $y = mx + b$ and noting that the slope is $\frac{11}{5}$. Then plugging in $(2, 2)$ for (x, y) we solve for b as $2 = \frac{11}{5}2 + b$ so $b = -\frac{12}{5}$. If we now change $y = 2.5$ we can solve for x and get

$$\begin{aligned} 2.5 &= \frac{11}{5}x - \frac{12}{5} \\ 2.5 + \frac{12}{5} &= \frac{11}{5}x \\ \frac{24.5}{11} &= x \end{aligned}$$

as our approximation.

3. [10 points] Answer the following problems True/False/Neither. Answer TRUE if it MUST BE true; answer FALSE if it MUST BE false; answer NEITHER otherwise. No explanations necessary, no partial credit. Your answer must be clear...if I cannot tell what your intended answer is, then it is wrong.

a. [2 points] If $\tan(x) = 3$ then $\arctan(x) = 1/3$.

Solution: FALSE

b. [2 points] $\frac{d}{dx}[x^x] = xx^{x-1}$

Solution: FALSE

c. [2 points] A continuous function always has a local maximum.

Solution: FALSE

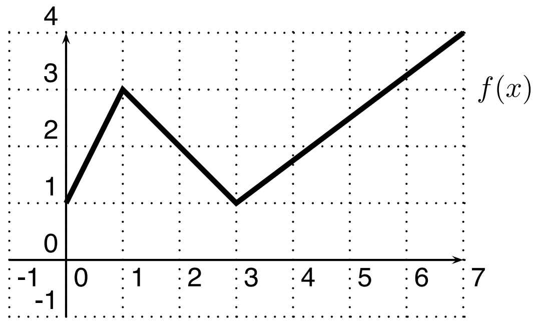
d. [2 points] The derivative of 2^x is $x2^{x-1}$.

Solution: FALSE

e. [2 points] A continuous function from the interval $[-2, 3]$ always has a global maximum.

Solution: TRUE

4. [13 points]



x	1	2	3	4	5	6
g(x)	0	4	0	-18	-56	-120
g'(x)	6	1	-10	-27	-50	-79
g"(x)	-2	-8	-14	-20	-26	-32

a. [6 points] Let $h(x) = \dfrac{g(x)}{f(2x+3)}$. Find $h'(1)$ or explain why it doesn't exist.

Solution: We use the quotient rule and get

$$h'(x) = \frac{g'(x)f(2x+3) - 2f'(2x+3)g(x)}{(f(2x+3))^2}.$$

If we plug in $x = 1$ we get

$$h'(1) = \frac{g'(1)f(5) - 2f'(5)g(1)}{(f(5))^2}.$$

So we need to figure out $f'(5)$. This is a straight line segment that is rising from $(3,1)$ to $(7,4)$ and so the slope is $(4-1)/(7-3) = 3/4$. We can also calculate $f(5) = 1 + 6/4$ because 5 is two more than 3 so the change in y is twice the slope.
Thus

$$h'(1) = \frac{6(10/4) - 2(3/4)(0)}{(10/4)^2} = 2.4.$$

b. [7 points] Let $k(x) = g(g(x))$. Determine whether $k(x)$ is increasing or decreasing at $x = 2$.

Solution: We again use the chain rule and get $k'(x) = g'(g(x)) * g'(x)$. Plugging in $x = 2$ gives us

$$k'(2) = g'(g(2)) * g'(2) = g'(4) * 1 = -27.$$

5. [14 points]
Let $f(x) = e^{ax^2+b}$, where a and b are constants.

a. [7 points] Find the critical points of $f(x)$. Your answer should depend on a and b.

Solution: If we differentiate our function to find the critical points, we get

$$f'(x) = 2axe^{ax^2+b}.$$

Since this always exists, the critical points are the zeroes, but this is clearly only when $x = 0$.

b. [7 points] Find the inflection points of $f(x)$. Your answer should depend on a and b.

Solution: We must find when the second derivative changes sign. We take the second derivative and get

$$f''(x) = 2ae^{ax^2+b} + 4a^2x^2e^{ax^2+b}.$$

Setting this equal to zero we get that $x = \pm\frac{1}{\sqrt{-2a^2}}$. But this square root never exists, and so the second derivative is never zero, and so there are no inflection points.

6. [14 points]
Consider all right triangles formed by the positive x and y axes and a line through the point $(4, 3)$. Which such triangle minimizes the length of the hypotenuse? (Hint: Draw a picture! Draw several...then label them with variables.)

Solution: We have the picture where we intersect at $x = a$ and $y = b$ and so the hypotenuse is $\sqrt{a^2 + b^2}$, so we want to minimize $x^2 + y^2$. Since the hypotenuse has to go through $(4, 3)$, our slope from this point to the x axis is $(-3)/(a-4)$ and our slope to the y axis is $(3-b)/(4)$. Since these slopes are equal we have that

$$-3/(a-4) = (3-b)/4.$$

And so $b = 12/(a-4) + 3$. Plugging this in, we now wish to minimize

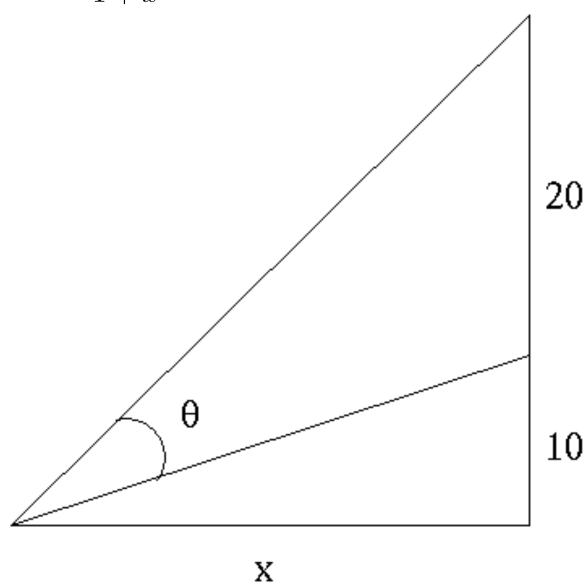
$$a^2 + \left(\frac{12}{a-4} + 3\right)^2.$$

Looking for the critical points, we find the derivative is

$$2a - \frac{24}{(a-4)^2}\left(\frac{12}{a-4} + 3\right).$$

Setting this equal to zero and solving gives us $a = 0$ or $a = 4 + \sqrt[3]{36}$. Since a cannot be less than 4, we have that $a = 4 + \sqrt[3]{36}$ and plugging in for b gives $b = 3 + 2\sqrt[3]{6}$.

7. [14 points]
A movie screen is 10 feet off the ground and is an additional 20 feet tall. How far from the screen should you seat yourself so that your viewing angle, θ, is as large as possible? (Hint: The derivative of $\arctan(x) = \frac{1}{1+x^2}$)



Solution: We see that the angle θ is the difference of the angle of the whole view minus the angle of the view of the base. This whole view has opposite side 30 and adjacent x, so $\arctan(30/x)$. The angle of the base is $\arctan(10/x)$. We wish to maximize their difference. This is

$$\arctan(30/x) - \arctan(10/x).$$

Taking the derivative yields

$$\frac{1}{1+(30/x)^2} * \frac{-30}{x^2} - \frac{1}{1+(10/x)^2} * \frac{-10}{x^2} = \frac{-30}{x^2+900} + \frac{10}{x^2+100}.$$

Setting this equal to zero and solving gives $x^2 = 300$, so $x = \pm 10\sqrt{3}$. But x cannot be negative. So $x = 10\sqrt{3}$. To check this is maximized we can use the first derivative test with $x = 10$ and $x = 1$ and see that it is indeed maximized.

8. [2 points] What is your favorite comedic movie?