Name: Solutions

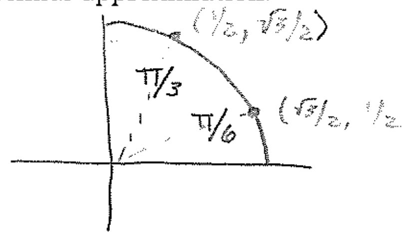
Math 105: Winter 2013
Exam 2: March 15

Correct answers accompanied by incorrect or incomplete work will not receive full credit.

1. (20 points) Let $f(x) = \arcsin \sqrt{x}$.

   (a) Find $f\left(\frac{3}{4}\right)$. Express your answer exactly, *not* as a decimal approximation.

$$\arcsin\left(\frac{\sqrt{3}}{2}\right) = \boxed{\pi/3}$$



   (b) Find $f'(x)$.

$$\frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2}x^{-1/2} = f'(x)$$

   (c) Find $f'\left(\frac{3}{4}\right)$. Express your answer ~~exactly, not~~ as a decimal approximation.

$$f'\left(\frac{3}{4}\right) = \frac{1}{\sqrt{1-3/4}} \cdot \frac{1}{2\sqrt{3/4}} = \frac{1}{\sqrt{1/4}} \cdot \frac{1}{2\sqrt{3/4}} = \sqrt{4} \cdot \frac{1}{2} \cdot \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

$$\boxed{f'\left(\frac{3}{4}\right) \approx 1.15}$$

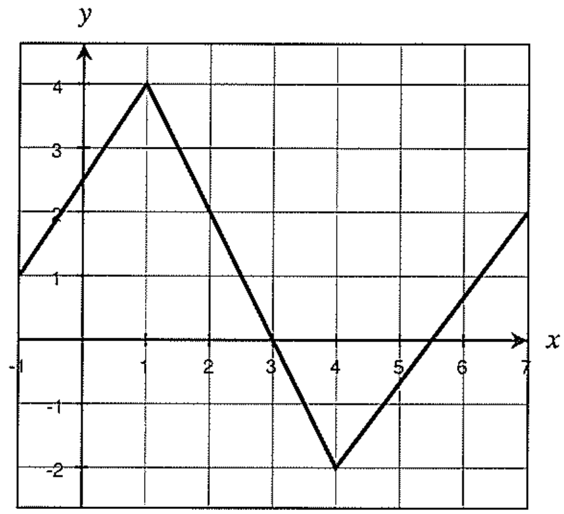
   (d) Write the equation of the line tangent to $f(x)$ at $x = \frac{3}{4}$.

goes through $\left(\frac{3}{4}, f\left(\frac{3}{4}\right)\right) = \left(\frac{3}{4}, \frac{\pi}{3}\right)$

w/ slope $= 1.15$

$$\boxed{y - \pi/3 = (1.15)\left(x - \frac{3}{4}\right)}$$

2. (18 points) Consider a function g that has the following graph.



Let $h(x) = \frac{x^{4/3}}{g(x)}$.

(a) Write an expression for $h'(x)$.

$$h'(x) = \frac{4/3\, x^{1/3} g(x) - x^{4/3} g'(x)}{(g(x))^2}$$

(b) Is h increasing at $x = 2$? Justify your answer using calculus.

$$h'(2) = \frac{4/3\,(2)^{1/3} g(2) - 2^{4/3} g'(2)}{(g(2))^2} \qquad g(2) = 2, \quad g'(2) = -2$$

$$= \frac{4/3\,(2)^{1/3}(2) - 2^{4/3}(-2)}{2^2} = \frac{8/3\,(2)^{1/3} + 2\,(2)^{4/3}}{4} = \text{positive}$$

$\boxed{\text{SO } h \text{ is increasing}}$

(c) For what values of x in the interval $[-1, 7]$ is $h'(x)$ undefined?

$h'(x)$ is undefined when the denominator is 0

i.e. when $g(x) = 0$, at $x = 3, 5.5$

It's also undefined when $g'(x)$ is undefined

i.e at the "kinks" $x = 1, 4$

so $\boxed{x = 1, 3, 4, 5.5}$

3. (16 points) Find y' in **2 of 3** of the following. If you do more than two, then clearly mark which two you want graded. If you don't, the **worst** two will be chosen for you.

(a) $y = 5e^{(x^3+2x)} + \sin^2 x + \cos\frac{\pi}{7}$

(b) $y = 3^x \arcsin x + \ln(\cos x) + e^7$

(c) $y = \arctan\left(x^3 + \cos\left(x^2 + 8x\right)\right)$

(a) $y' = 5e^{(x^3+2x)}(3x^2+2) + 2\sin x \cos x$

(b) $y' = 3^x \ln 3 \arcsin x + 3^x \frac{1}{\sqrt{1-x^2}} + \frac{1}{\cos x}(-\sin x)$

(c) $y' = \frac{1}{1+(x^3+\cos(x^2+8x))^2}\left[3x^2 - (\sin(x^2+8x))(2x+8)\right]$

4. (8 points) Find $\frac{dy}{dx}$ in **1 of 2** of the following. If you do more than one, then clearly mark which one you want graded. If you don't, the **worst** will be chosen for you.

(a) $y = (\cos x)^x$

(b) $\cos y + xy^2 = 15y$

(a) $\ln y = x \ln(\cos x)$

$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{\cos x}(-\sin x) + \ln(\cos x)$

$\frac{dy}{dx} = (\cos x)^x \left[\frac{-x\sin x}{\cos x} + \ln(\cos x)\right]$

(b) $-\sin y \frac{dy}{dx} + x\, 2y \frac{dy}{dx} + y^2 = 15\frac{dy}{dx}$

$(-\sin y + 2xy - 15)\frac{dy}{dx} = -y^2$

$\frac{dy}{dx} = \frac{-y^2}{-\sin y + 2xy - 15}$

5. (8 points) Find the solution to the Initial Value Problem

$$y' = \frac{9}{x} + 2^x + \cancel{5}x^2 + 17, \quad y(1) = 5.$$

6

$y = \ln|x| + \frac{2^x}{\ln 2} + \frac{6}{3}x^3 + 17x + C$

$5 = \ln\cancel{|1|} + \frac{2^1}{\ln 2} + \cancel{\frac{6}{3}} + 17 + C$

0 2

$-14 - \frac{2}{\ln 2} = C$

$C \approx -16.89$

$y = \ln|x| + \frac{2^x}{\ln 2} + 2x^3 + 17x - 14 - \frac{2}{\ln 2}$

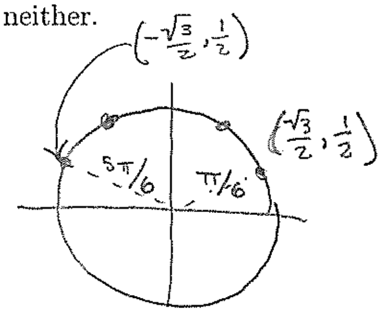
6. (12 points) Let f be a function with first and second derivatives given below:

$$f'(x) = \frac{\sin x - \frac{1}{2}}{x-2}, \qquad f''(x) = \frac{(x-2)\cos x - (\sin x - \frac{1}{2})}{(x-2)^2}$$

Find the *critical points* of f on the interval $[0, 2\pi]$. Give the x-values exactly, *not* as decimal approximations. Classify each critical point as a local maximum, local minimum, or neither.

$f'(x) = 0 \implies \sin x - \frac{1}{2} = 0 \implies \sin x = \frac{1}{2}$

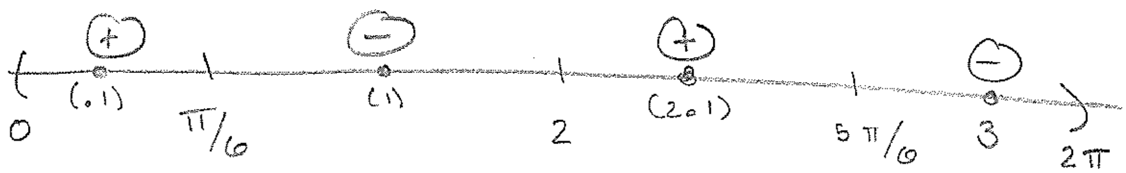
$x = \pi/6,\ 5\pi/6$



$f'(x)$ undefined $\implies x - 2 = 0 \implies x = 2$

Three critical points $\boxed{x = \pi/6 \approx .52,\ 2,\ 5\pi/6 \approx 2.62}$

$f'(x)$



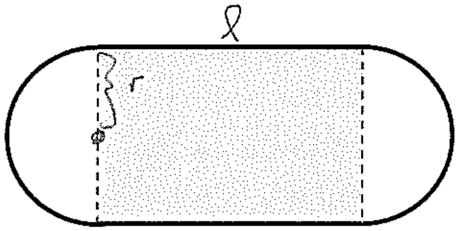
$f'(.1) = +,\quad f'(1) = -,\quad f'(2.1) = +,\quad f'(3) -$

$f(x)$

so $x = \pi/6$ and $x = 5\pi/6$ are local maxs

and $x = 2$ is a local min.

7. (18 points) An outdoor track is to be created in the shape of a rectangle with semicircles attached on two opposite ends of the rectangle. The track must have a perimeter of 440 yards. Find the dimensions for the track that maximize the area of the *rectangular* (shaded) portion of the field enclosed by the track. (Helpful formulas related to circles: Area $= \pi r^2$, Circumference $= 2\pi r$.)



maximize area of shaded region

(a) What is the objective function for the quantity you are trying to optimize?

$A = 2r\ell$

(b) What is the constraint equation?

$P = 440 = 2\ell + 2\pi r$

(c) Answer the question. (Be sure you use calculus to verify that your critical point optimizes the objective function.)

$440 = 2\ell + 2\pi r$

$440 - 2\pi r = 2\ell \longrightarrow \ell = 220 - \pi r$

$A = 2r(220 - \pi r) = 440r - 2\pi r^2$

$A' = 440 - 4\pi r,\quad 440 - 4\pi r = 0$

$440 = 4\pi r$

$\frac{110}{\pi} = r \approx 35.01$

$A'' = -4\pi$ = neg, so $r \approx 35.01$ maximizes A

$\ell = 220 - \pi\left(\frac{110}{\pi}\right) = 110$

So a track w/ $r = \frac{110}{\pi}$ yds and $\ell = 110$ yds maximizes the shaded area.