TEST 2 SOLUTIONS

Math 105
3/14/14

Name: ______________________
by writing my name I swear this work is my own

Read all of the following information before starting the exam:

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 8 problems and is worth 100 points, It is your responsibility to make sure that you have all of the pages!
- Good luck!
- Some useful formulas:
  Area of a circle with radius r, $A = \pi r^2$,
  Circumference of a circle with radius r, $C = 2\pi r$.
  Area of a rectangle with sides x and y, $A = xy$,
  Perimeter of a rectangle with sides x, y, $P = 2x + 2y$
  Volume of a rectangular prism with sides x, y, z, $V = xyz$,
  Surface area of rectangular prism with x, y, z, $SA = 2xy + 2yz + 2xz$
  Volume of a cylinder with radius r and height h, $V = \pi r^2 h$,
  Surface area of a cylinder with radius r and height h, $SA = 2\pi r^2 + 2\pi rh$

1. (*14 points*)

x	$f(x)$	$g(x)$	$j(x)$	$f'(x)$	$g'(x)$	$j'(x)$
-2	0	1	-1	3	2	1
-1	1	3	2	-1	3	0
0	2	1	1	2	-2	2
1	3	1	-1	1	3	1
2	-2	2	1	3	0	3
3	-1	1	-1	1	-2	2

a. (*7 pts*) $H(x) = f(g(x)) + \dfrac{j(x)}{x+1}$. Find $H'(2)$.

$$H'(x) = f'(g(x)) * g'(x) + \frac{j'(x)(x+1) - j(x)}{(x+1)^2}$$

$$H'(2) = f'(2) * 0 + \frac{3 * 3 - 1}{9} = \frac{8}{9}$$

b. (*7 pts*) $F(x) = g(x)^3 - x^2 f(x)$. Find $F'(1)$.

$$F'(x) = 3g(x)^2 * g'(x) - (2xf(x) + x^2 f'(x)$$

$$F'(1) = 3 * 1 * 3 - (2 * 3 + 1 * 1) = 9 - 7 = 2$$

2. (*8 points*) Find $f'(x)$ using logarithmic differentiation.

$$f(x) = \frac{\cos^4(x)(10x^3 - 2x)^3}{e^{x^2}\sin(2x)}$$

$$\ln(f(x)) = 4\ln(\cos(x)) + 3\ln(10x^3 - 2x) - x^2 - \ln(sin(2x))$$

$$f'(x) = \left(\frac{-4\sin(x)}{\cos(x)} + \frac{90x^2 - 6}{10x^3 - 2x} - 2x - \frac{2cos(2x)}{sin(2x)}\right)\frac{\cos^4(x)(10x^3 - 2x)^3}{e^{x^2}\sin(2x)}$$

3. (*15 points*) Find $f'(x)$.

a. (*9 pts*) $f(x) = \sqrt[3]{(\cos^2(x) + 3 + \sin^2(x))} + \arcsin\left(\frac{2}{x}\right) + xe^{\tan(x)}$

$$\frac{1}{3}(\cos^2(x)+3+\sin^2(x))^{-2/3}(2\cos(x)*(-\sin(x))+2\sin(x)*cos(x))+\frac{1}{\sqrt{1-\left(\frac{2}{x}\right)^2}}\left(\frac{-2}{x^2}\right)+e^{\tan(x)}+xe^{\tan(x)}\sec^2(x)$$

Note, the term $(2\cos(x) * (-\sin(x)) + 2\sin(x) * cos(x)) = 0$, so really $(\sqrt[3]{(\cos^2(x) + 3 + \sin^2(x))})' = 0$

$$\frac{1}{\sqrt{1-\left(\frac{2}{x}\right)^2}}\left(\frac{-2}{x^2}\right) + e^{\tan(x)} + xe^{\tan(x)}\sec^2(x)$$

b. (*6 pts*) $f(x) = 2^{3x}\cos(4x) + \ln(\sin(2x))$

$$\ln(2)2^{3x} * 3\cos(4x) - 2^{3x}\sin(4x) * 4 + \frac{1}{\sin(2x)} * \cos(2x) * 2$$

4. (*16 points*)

a. (*8 pts*) For the equation $e^{\cos(y)} = x^4\arctan(y)$ use implicit differentiation to find $\frac{dy}{dx}$.

$$e^{\cos(y)} * (-\sin(y)) * y' = 4x^3\arctan(y) + x^4\frac{1}{1+y^2}y\prime$$

$$y' = \frac{4x^3\arctan(y)}{e^{\cos(y)} * (-\sin(y)) - x^4\frac{1}{1+y^2}}$$

b. (*8 pts*) <u>**Let a be a constant**</u>, find the equation of the tangent line at $(a, 0)$ of

$$x^{2/3} + y^{2/3} = a^{2/3}$$

$$2/3x^{-1/3} + 2/3y^{-1/3}y' = 0$$

$$y' = \frac{-x^{-1/3}}{y^{-1/3}} = \frac{y^{1/3}}{x^{1/3}}$$

So, the line is $y = 0$.

5. (*15 points*) Find the antiderivative of the given function.

a. (*5 pts*) $f(x) = \dfrac{1}{\sqrt{1-4x^2}}$.

$$F(x) = \frac{1}{2}\arcsin(2x) + C$$

b. (*5 pts*) $g(x) = \dfrac{\arctan(x)}{1+x^2}$.

$$F(x) = \frac{1}{2}\arctan(x)^2 + C$$

c. (*5 pts*) $h(x) = \dfrac{1}{(\arctan(x))(1+x^2)}$.

$$H(x) = \ln(\arctan(x)) + C$$

6. (*14 points*) Evaluate the following limits. Only use L'Hôpital's rule when appropriate. Show your work!!

a. (*6 pts*) $\displaystyle\lim_{x\to 0} \frac{x-\sin(x)}{x^2}$

$\displaystyle\lim_{x\to 0} \frac{x-\sin(x)}{x^2}$ is Type $\frac{0}{0}$

$$\lim_{x\to 0} \frac{x-\sin(x)}{x^2} = \lim_{x\to 0} \frac{1-\cos(x)}{2x}$$

This is again Type $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{1-\cos(x)}{2x} = \lim_{x\to 0} \frac{\sin(x)}{2} = 0$$

b. (*8 pts*) $\displaystyle\lim_{x\to\infty} x^{\dfrac{\ln(k)}{\ln(x)+1}}$, for k a constant.

This limit is Type ∞^0

$$y = \lim_{x\to\infty} x^{\dfrac{\ln(k)}{\ln(x)+1}}$$

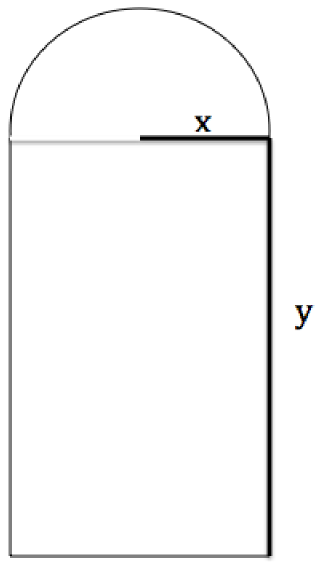
$$ln(y) = \lim_{x\to\infty} \frac{\ln(k)}{\ln(x)+1}\ln(x) = \lim_{x\to\infty} \frac{\ln(k)ln(x)}{\ln(x)+1}$$

This is now Type $\frac{\infty}{\infty}$.

$$ln(y) = \lim_{x\to\infty} \frac{\ln(k)ln(x)}{\ln(x)+1} = \lim_{x\to\infty} \frac{\ln(k)\frac{1}{x}}{\frac{1}{x}} = \ln(k)$$

So $y = k$ and $\displaystyle\lim_{x\to\infty} x^{\dfrac{\ln(k)}{\ln(x)+1}} = k$

7. (*14 points*) Consider the following window formed by a rectangle and a semi-circle. There is 50 feet of metal piping to go around the perimeter of the window. What are the dimensions (x and y) to maximize the area of the window?



Objective: $A = 2xy + \frac{1}{2}\pi x^2$

Constraint: $50 = 2x + 2y + \pi x$

So, $y = \frac{50-2x-\pi x}{2}$ and $A = 2x\frac{50-2x-\pi x}{2} + \frac{1}{2}\pi x^2 = 50x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2 = 50x - 2x^2 - \frac{1}{2}\pi x^2$.

$$A' = 50 - 4x - \pi x$$

So, $A' = 0$ when $x = 50/(4+\pi) \cong 7$ is a critical point.

$$A'' = -4 - \pi < 0$$

So, $x = 50/(4+\pi)$ ft is a max.

Then, $y = \frac{50-2(50/(4+\pi))-\pi(50/(4+\pi))}{2} \cong 7$

The dimensions make the bottom part a square.

8. (*4 points*) For the following optimization problem, draw a picture, find a function for the objective function, and find a function for the constraint function. DO NOT SOLVE.

A kaleidoscope is a cylinder with a glass top, glass base, and cardboard all around. If the cardboard costs \$2 per sq. ft and the glass costs \$5 per sq.ft, then what dimensions (radius and height) maximize the volume of the kaleidoscope if we only have \$20 to spend?

::Picture of a cylinder with radius and height marked::

Objective: $V = \pi r^2 h$

Constraint: $20 = 5(2\pi r^2) + 2(2\pi rh) = 10\pi r^2 + 4\pi rh$

9. (*2 points*) BONUS: Who discovered L'Hôpital's rule?

Bernoulli