

Math 105: Review for Exam II - Solutions

1. Find dy/dx for each of the following.

(a) $y = x^2 + 2^x + e^2 + e^{2x} + \ln 2 + \ln(2x) + \arctan 2$

$$\frac{dy}{dx} = 2x + (\ln 2)2^x + 2e^{2x} + \frac{1}{2x} \cdot 2 \quad \text{Note that } e^2, \ln 2, \text{ and } \arctan 2 \text{ are constants.}$$

(b) $y = \sqrt{x} \cdot \arctan(5x)$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} \arctan(5x) + \sqrt{x} \cdot \frac{1}{1+(5x)^2} \cdot 5 = \frac{\arctan(5x)}{2x^{1/2}} + \frac{5\sqrt{x}}{1+25x^2}$$

(c) $y = \ln(\tan(2^{\cos(x^2)}))$

$$\frac{dy}{dx} = \frac{1}{\tan(2^{\cos(x^2)})} \cdot \sec^2(2^{\cos(x^2)}) \cdot \ln 2(2^{\cos(x^2)}) \cdot (-\sin(x^2)) \cdot 2x$$

(d) $y = \frac{x + e^\pi}{\cos 4 + \sin^5(6x)}$

Note that e^π and $\cos 4$ are constants.

$$\frac{dy}{dx} = \frac{(1)(\cos 4 + \sin^5(6x)) - (x + e^\pi)(5 \sin^4(6x) \cdot \cos(6x) \cdot 6)}{(\cos 4 + \sin^5(6x))^2} \quad \text{Recall that } \sin^5(6x) = (\sin(6x))^5.$$

2. Consider the curve defined by $x^3 + y^3 = \frac{9}{2}xy$ (known as the Folium of Descartes).

(a) Find dy/dx .

Use implicit differentiation.

$$\begin{aligned} 3x^2 + 3y^2 \frac{dy}{dx} &= \frac{9}{2}y + \frac{9}{2}x \frac{dy}{dx} \\ 3y^2 \frac{dy}{dx} - \frac{9}{2}x \frac{dy}{dx} &= \frac{9}{2}y - 3x^2 \\ \frac{dy}{dx} \left(3y^2 - \frac{9}{2}x \right) &= \frac{9}{2}y - 3x^2 \\ \frac{dy}{dx} &= \frac{\frac{9}{2}y - 3x^2}{3y^2 - \frac{9}{2}x} \end{aligned}$$

(b) Verify that the point (1,2) is on the curve above.

We must check to see if the values $x = 1$ and $y = 2$ satisfy the equation above.

$$\begin{aligned} x^3 + y^3 &\stackrel{?}{=} \frac{9}{2}xy \\ 1^3 + 2^3 &\stackrel{?}{=} \frac{9}{2} \cdot 1 \cdot 2 \\ 9 &\stackrel{?}{=} 9 \end{aligned}$$

Thus, the point (1,2) is on the curve.

(c) Find the equation of the tangent line at the point (1,2).

We want $y = mx + b$.

$$m = \frac{\frac{9}{2} \cdot 2 - 3 \cdot 1^2}{3 \cdot 2^2 - \frac{9}{2} \cdot 1} = \frac{4}{5}, \text{ so } y = \frac{4}{5}x + b.$$

Now plug in $x = 1$ and $y = 2$ to find b .

$$2 = \frac{4}{5} \cdot 1 + b \Rightarrow \frac{6}{5} = b$$

Therefore, we have $y = \frac{4}{5}x + \frac{6}{5}$.

3. Evaluate the following limits.

Throughout this solution, the symbol ★ will stand for whatever notation your instructor prefers for using L'Hopital's Rule on the indeterminate form $0/0$; this may be $\stackrel{0/0}{=}$ or $\stackrel{L'H}{=}$ or $\stackrel{H}{=}$ or $\stackrel{0/0}{=}$ or "has the form ' $\frac{0}{0}$ ', and so, by L'Hopital's Rule, is equal to" or something else. The symbol ♡ will serve the same purpose for the indeterminate forms ∞/∞ and $-\infty/\infty$.

$$(a) \lim_{x \rightarrow 1} \frac{x^3 - 1}{7 - 7x} \star \lim_{x \rightarrow 1} \frac{3x^2}{-7} = \frac{3}{-7} = -\frac{3}{7}$$

$$(b) \lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{3^x} = \frac{0}{1} = 0$$

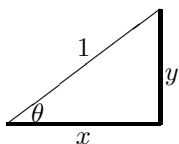
Can't use (and don't need) L'Hopital's Rule!

$$(c) \lim_{x \rightarrow 0} \frac{1 - \cos(4x)}{5x^2} \star \lim_{x \rightarrow 0} \frac{4 \sin(4x)}{10x} \star \lim_{x \rightarrow 0} \frac{16 \cos(4x)}{10} = \frac{16}{10} = \frac{8}{5}$$

$$(d) \lim_{x \rightarrow \infty} \frac{x^2}{2^x} \heartsuit \lim_{x \rightarrow \infty} \frac{2x}{\ln 2 \cdot 2^x} \heartsuit \lim_{x \rightarrow \infty} \frac{2}{\ln 2 \cdot \ln 2 \cdot 2^x} = 0$$

4. Rewrite $\tan(\arccos x)$ as an algebraic expression - no trigonometric or inverse trigonometric functions. [Students in the 8:00 section may omit this problem.]

Let $\theta = \arccos x$. That is, θ is the angle whose cosine is x .



$$x^2 + y^2 = 1^2 \Rightarrow y = \sqrt{1 - x^2}$$

$$\tan(\arccos x) = \tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} = \frac{\sqrt{1 - x^2}}{x}$$

5. Consider the function $f(x) = x^4 e^x$ with domain all real numbers.

(a) Find the x -value(s) of all roots (x -intercepts) of f .

The equation $x^4 e^x = 0$ means $x^4 = 0$ (that is, $x = 0$) or $e^x = 0$ (no solution), so the only root is at $x = 0$.

(b) Find the x - and y -value(s) of all critical points and identify each as a local max, local min, or neither.

$$\begin{aligned} f'(x) &= 4x^3 e^x + x^4 e^x \\ 0 &= x^3 e^x (4 + x) \\ \Rightarrow x &= 0, -4 \end{aligned}$$

Note that e^x is never 0.

	$x < -4$	$-4 < x < 0$	$4 < x$
f'	positive	negative	positive
f	↗	↘	↗

$$y\text{-values: } f(-4) = 256e^{-4} \approx 4.689, f(0) = 0$$

So, f has a local maximum at $(-4, 256e^{-4})$ and a local minimum at $(0, 0)$.

(c) Find the x - and y -value(s) of all global extrema and identify each as a global max or global min.

There is a global minimum at $(0, 0)$. There is no global maximum because as $x \rightarrow \infty$, $f(x) \rightarrow \infty$. Note that as $x \rightarrow -\infty$, $f(x) \rightarrow 0$. You can verify this by using L'Hopital's Rule on x^4/e^{-x} .

(d) Find the x -value(s) of all inflection points.

$$f''(x) = 12x^2e^x + 4x^3e^x + 4x^3e^x + x^4e^x \quad \text{Use Product Rule on each product in } f'(x) \text{ above.}$$

$$0 = e^x(x^4 + 8x^3 + 12x^2)$$

$$0 = e^x x^2(x^2 + 8x + 12)$$

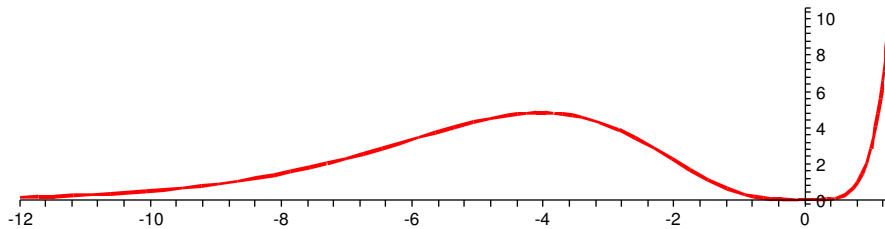
$$0 = e^x x^2(x + 2)(x + 6)$$

$$\Rightarrow x = 0, -2, -6$$

	$x < -6$	$-6 < x < -2$	$-2 < x < 0$	$0 < x$
f''	positive	negative	positive	positive
f	concave up	concave down	concave up	concave up

So, the x -values of the inflection points of f are $x = -2$ and $x = -6$ but NOT $x = 0$.

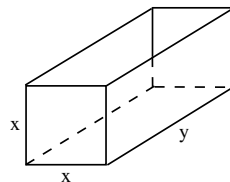
(e) Sketch f .



6. How would your answers to the previous question change if the domain of f were $[-10, 10]$?

There would be a global maximum at $(10, 10^4e^{10})$. (And the graph would be restricted to $-10 \leq x \leq 10$).

7. You are planning to build a box-shaped aquarium with no top and with two square ends. Your budget is \$288. If the glass for the sides costs \$12 per square foot and the opaque material for the bottom costs \$3 per square foot, what dimensions will maximize the volume? Be sure to show how you know you have found the maximum.



Goal: Maximize volume

Objective function: volume = $V = x \cdot x \cdot y = x^2y$

We need to get this down to a function of just one variable, so we use the *constraint equation*:

$$\text{total cost} = (\text{cost of base}) + (\text{cost of two square ends}) + (\text{cost of two other sides})$$

$$288 = 3xy + 12 \cdot 2x^2 + 12 \cdot 2xy$$

$$288 = 27xy + 24x^2$$

$$288 - 24x^2 = 27xy$$

$$\frac{288 - 24x^2}{27x} = y$$

Substituting this back into the objective function gives

$$V = x^2y = x^2 \cdot \frac{288 - 24x^2}{27x} = x \cdot \frac{288 - 24x^2}{27} = \frac{1}{27}(288x - 24x^3).$$

Now that we have V as a function of just one variable, we find its maximum.

$$V'(x) = \frac{1}{27}(288 - 72x^2)$$

$$0 = \frac{1}{27}(288 - 72x^2)$$

$$0 = (288 - 72x^2)$$

$$72x^2 = 288$$

$$x^2 = \frac{288}{72}$$

$$x = 2$$

We discard $x = -2$ because lengths must be nonnegative.

Since V' is positive for $x < 2$ and negative for $2 < x$, we know that the maximum occurs at $x = 2$.

And $y = \frac{288 - 24x^2}{27x} = \frac{288 - 24 \cdot 2^2}{27 \cdot 2} = \frac{32}{9}$, so the dimensions are 2 by 2 by $\frac{32}{9}$.

8. Use the Intermediate Value Theorem to explain why $f(x) = x^3 - 4x^2 + 5$ must have a root somewhere on the interval $[1, 2]$.

IVT: If f is continuous on $[a, b]$ and y is a number between $f(a)$ and $f(b)$, then there is a number c between a and b such that $f(c) = y$.

Our function f is continuous on $[1, 2]$. We can compute that $f(1) = 2$ and $f(2) = -3$. Since 0 is a number between 2 and -3 , the IVT says there is a number c between 1 and 2 such that $f(c) = 0$; this c is the desired root.

[In plainer English, f is positive at one endpoint and negative at the other. Since f is continuous, the only way its value can go from positive to negative is to go through zero; where f is zero is our root.]

9. Let $y = \frac{x^3 \cos(x)}{x^2 + 1}$.

- (a) Find $\frac{dy}{dx}$ using the product and quotient rules.

$$\frac{dy}{dx} = \frac{(3x^2 \cos(x) + (-\sin(x))x^3)(x^2 + 1) - 2x(x^3 \cos(x))}{(x^2 + 1)^2}$$

- (b) Find $\frac{dy}{dx}$ using logarithmic differentiation. [Students in the 1:10 section may consider this as a bonus problem.]

$$\ln y = \ln \left[\frac{x^3 \cos(x)}{x^2 + 1} \right]$$

Take \ln of each side.

$$\ln y = 3 \ln x + \ln(\cos(x)) - \ln(x^2 + 1)$$

Apply log rules.

$$\frac{1}{y} \frac{dy}{dx} = 3 \frac{1}{x} + \frac{1}{\cos(x)}(-\sin(x)) - \frac{1}{x^2 + 1}(2x)$$

Differentiate each side.

$$\frac{dy}{dx} = \left[\frac{3}{x} - \frac{\sin(x)}{\cos(x)} - \frac{2x}{x^2 + 1} \right] \cdot y$$

$$\frac{dy}{dx} = \left[\frac{3}{x} - \frac{\sin(x)}{\cos(x)} - \frac{2x}{x^2 + 1} \right] \cdot \frac{x^3 \cos(x)}{x^2 + 1}$$

Do the two methods give the same answer? They should! Check for yourself.