MATH106A CALCULUS II - PROF. P. WONG

EXAM II - MARCH 13, 2015

NAME:

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

Advice: DON'T spend too much time on a single problem.

Problems	**Maximum Score**	**Your Score**
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
Total	100	

1. Evaluate each of the following indefinite integrals (be sure to indicate what techniques you use).

(10 pts.)(a)

$$\int \cos^2 x \sin x \, dx.$$

Let $u = \cos x$ **so that** $du = -\sin x \, dx$**. It follows that**

$$\int \cos^2 x \sin x \, dx = -\int u^2 \, du = -\frac{u^3}{3} + C$$
$$= -\frac{\cos^3 x}{3} + C.$$

(10 pts.)(b)

$$\int x \cos(3x) \, dx.$$

Let $u = x$ **and** $dv = \cos(3x) \, dx$**. Then,** $du = dx$ **and** $v = \dfrac{\sin(3x)}{3}$**. It follows from the technique of integration by parts that**

$$\int x \cos(3x) \, dx \overset{\text{IBP}}{=} x \cdot \frac{\sin(3x)}{3} - \int \frac{1}{3} \sin(3x) \, dx$$
$$= \frac{x}{3} \sin(3x) + \frac{1}{9} \cos(3x) + C.$$

2. Evaluate each of the following indefinite integrals (be sure to indicate what techniques you use).

(10 pts.)(a)

$$\int \frac{x^3+1}{x^2-4}\,dx.$$

Since the numerator has higher degree than that of the denominator, we use long division to write the rational function as

$$\frac{x^3+1}{x^2-4} = x + \frac{4x+1}{x^2-4} = x + \frac{4x+1}{(x+2)(x-2)}.$$

Write $\frac{4x+1}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$ **so that**

$$4x+1 \equiv A(x-2)+B(x+2).$$

At $x=2$**, we have** $9=4B$ **or** $B=\frac{9}{4}$**. At** $x=-2$**, we have** $-7=-4A$ **or** $A=\frac{7}{4}$**. Thus,**

$$\frac{x^3+1}{x^2-4} = x + \frac{7}{4}\cdot\frac{1}{x+2} + \frac{9}{4}\cdot\frac{1}{x-2}.$$

Hence,

$$\int \frac{x^3+1}{x^2-4}\,dx = \frac{x^2}{2} + \frac{7}{4}\ln|x+2| + \frac{9}{4}\ln|x-2| + C.$$

(10 pts.)(b)

$$\int \frac{1}{(x^2+1)^{3/2}}\,dx.$$

Let $x=\tan\theta$ **so that** $dx=\sec^2\theta\;d\theta$**. It follows that**

$$\begin{aligned}
\int \frac{1}{(x^2+1)^{3/2}}\,dx &= \int \frac{\sec^2\theta\;d\theta}{(\tan^2\theta-1)^{3/2}} = \int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta \\
&= \int \frac{1}{\sec\theta}\,d\theta = \int \cos\theta\;d\theta \\
&= \sin\theta + C \quad (\textbf{now use the triangle of substitution}) \\
&= \frac{x}{\sqrt{1+x^2}} + C.
\end{aligned}$$

3. Evaluate each of the following improper integrals.

(10 pts.)(a)

$$\int_0^\infty \frac{x\ dx}{(x^2+1)^2}$$

First, $\displaystyle\int_0^\infty \frac{x\ dx}{(x^2+1)^2} = \lim_{b\to\infty}\int_0^b \frac{x\ dx}{(x^2+1)^2}$**. Now, let** $u = x^2+1$ **so** $du = 2x\ dx$**. It follows that**

$$\int \frac{x}{(x^2+1)^2}\ dx = \frac{1}{2}\int \frac{du}{u^2} = -\frac{1}{2u} + C$$
$$= -\frac{1}{2(x^2+1)} + C.$$

Now,

$$\int_0^\infty \frac{x\ dx}{(x^2+1)^2} = \lim_{b\to\infty}\int_0^b \frac{x\ dx}{(x^2+1)^2} = \lim_{b\to\infty} -\frac{1}{2(b^2+1)} + \frac{1}{2} = \frac{1}{2}.$$

(10 pts.)(b)

$$\int_0^2 \frac{dx}{(x-1)^{2/3}}$$

First note that this integral is *improper* **since** $\frac{1}{(x-1)^{2/3}}$ **is not defined at** $x=1$**. Thus, we rewrite the integral as**

$$\int_0^2 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^2 \frac{dx}{(x-1)^{2/3}}$$
$$= \lim_{b\to1}\int_0^b \frac{dx}{(x-1)^{2/3}} + \lim_{c\to1}\int_c^2 \frac{dx}{(x-1)^{2/3}}.$$

A simple substitution with $u = x-1$ **shows that** $\int \frac{dx}{(x-1)^{2/3}} = \int u^{-2/3}\ du = 3u^{1/3} + C = 3(x-1)^{1/3} + C$**. Now,**

$$\lim_{b\to1}\int_0^b \frac{dx}{(x-1)^{2/3}} = \lim_{b\to1}(3(b-1)^{1/3} - 3(-1)^{1/3}) = 3$$

and

$$\lim_{c\to1}\int_c^2 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to1}(3(1)^{1/3} - 3(c-1)^{1/3}) = 3.$$

Therefore,

$$\int_0^2 \frac{dx}{(x-1)^{2/3}} = 3+3 = 6.$$

4. Let $f(x) = \frac{1}{2x-1}$.

(8 pts.)(a) Find the third-order Taylor polynomial $P_3(x)$ of $f(x)$ centered at $x_0 = 1$.

Since $f(x) = (2x-1)^{-1}$**, we have** $f'(x) = -2(2x-1)^{-2}, f''(x) = 8(2x-1)^{-3}$ **and** $f'''(x) = -48(2x-1)^{-4}$**. It follows that** $f(1) = 1, f'(1) = -2, f''(1) = 8$ **and** $f'''(1) = -48$**. Now, the third-order Taylor polynomial is given by**

$$\begin{aligned} P_3(x) &= 1 + (-2)(x-1) + \frac{(8)}{2!}(x-1)^2 + \frac{(-48)}{3!}(x-1)^3 \\ &= 1 - 2(x-1) + 4(x-1)^2 - 8(x-1)^3. \end{aligned}$$

(6 pts.)(b) Find the third-order Maclaurin polynomial $M_3(x)$ of $f(x)$.

Using (a), with $x_0 = 0$**, we have** $f(0) = -1, f'(0) = -2, f''(0) = -8, f'''(0) = -48$**. Thus the third-order Maclaurin polynomial is given by**

$$\begin{aligned} M_3(x) &= (-1) + (-2)x + \frac{(-8)}{2!}x^2 + \frac{(-48)}{3!}x^3 \\ &= -1 - 2x - 4x^2 - 8x^3. \end{aligned}$$

(6 pts.)(c) What is the maximum error committed by using $M_3(x)$ (as in part (b)) over the interval $[1, 2]$, according to Taylor's Theorem? [Hint: how do you obtain K_4?]

To obtain K_4**, first note that** $f^{(4)}(x) = 384(2x-1)^{-5}$**. Over the interval** $[1.2]$**,** $|f^{(4)}(x)| \leq 385(2-1)^{-5} = 384$**. Thus, choose** $K_4 = 384$**. According to Taylor's theorem, we have**

$$|f(x) - M_3(x)| \leq \frac{K_4}{4!}|x-0|^4 \leq \frac{384}{4!} \cdot 2^4 = 256.$$

5. (12 pts.)(a) Use comparison to determine whether the following improper integral converges or diverges. Justify your answer.

$$\int_4^\infty \frac{1}{x^2 \ln x}\, dx$$

For $x \geq 4$**,** $\ln x > 1$ **so** $x^2 \ln x > x^2$**. It follows that** $\frac{1}{x^2 \ln x} < \frac{1}{x^2}$ **so that**

$$0 < \int_4^\infty \frac{1}{x^2 \ln x}\, dx < \int_4^\infty \frac{1}{x^2}\, dx < \int_1^\infty \frac{dx}{x^2}$$

which converges by the p**-test with** $p = 2 > 1$**. Thus we conclude that** $\int_4^\infty \frac{1}{x^2 \ln x}\, dx$ **converges.**

(8 pts.)(b) Consider the following function

$$f(x) = \begin{cases} \frac{C}{(x+1)^3}, & \text{for } x \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

For what value of C is $f(x)$ a *probability density function*?

For f **to be a p.d.f.,** $f(x) \geq 0$ **and** $\int_{-\infty}^\infty f(x)\, dx = 1$**. So** $C \geq 0$**. Note that**

$$\begin{aligned} \int_{-\infty}^\infty f(x)\, dx &= \int_0^\infty \frac{C}{(x+1)^3}\, dx = \lim_{b\to\infty} \int_0^b \frac{C}{(x+1)^3}\, dx \\ &= \lim_{b\to\infty} -\frac{C}{2}(x+1)^{-2}\Big|_0^b = -\frac{C}{2} \lim_{b\to\infty} [(b+1)^{-2} - (1)^{-2}] \\ &= -\frac{C}{2}[0-1] = \frac{C}{2}. \end{aligned}$$

The condition $\int_{-\infty}^\infty f(x)\, dx = 1$ **implies that** $\frac{C}{2} = 1$ **or** $C = 2$**.**