Math 106 Winter 2015

Test 2 (50 points)

Name: Solutions.

Show all your work to receive full credit for a problem. Points will be taken off if you do not show how you arrived at your answer, even if the final answer is correct.

Please keep your written answers brief; be clear and to the point. Points will be taken off for rambling and for incorrect or irrelevant statements.

Do not use the calculator integral function. Whenever possible, find the exact values of integrals by finding antiderivatives or using the table of integrals.

When you use a formula from the table of integrals, mention the formula number and the value(s) of any constant(s) that you may need.

Give exact answers. If needed, round off your answers to four decimal places.

Include units in your answers wherever possible.

There are seven questions. Questions are printed on both sides of a page.

You may use any of the following facts:

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}\,|x - x_0|^{n+1}$$

$$\int u\,dv = uv - \int v\,du$$

$\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$ and diverges for $p \leq 1$.

$\displaystyle\int_0^\infty e^{-ax}\,dx$ converges for $a > 0$.

1. (7 points) Evaluate the following integral. (You may use formulas 1-18, 39-42, 50, 51 only from the table of integrals for this problem.)

$$\int x^2 \cos x \, dx$$

$u = x^2 \qquad dv = \cos x \, dx$

$du = 2x\,dx \qquad v = \int \cos x \, dx = \sin x.$

$\int x^2 \cos x \, dx = uv - \int v\,du$

$= x^2 \sin x - \int \sin x . 2x\,dx$

$= x^2 \sin x - 2\int x \sin x \, dx.$

$u = x \qquad dv = \sin x \, dx$

$du = dx \qquad v = \int \sin x \, dx = -\cos x.$

So $\int x \sin x \, dx = x(-\cos x) - \int -\cos x \, dx$

$= -x\cos x + \int \cos x \, dx$

$= -x\cos x + \sin x.$

$\int x^2 \cos x \, dx = x^2 \sin x - 2\int x \sin x \, dx$

$= x^2 \sin x - 2(-x\cos x + \sin x)$

$= x^2 \sin x + 2x\cos x - 2\sin x + C.$

2. (7 points) Evaluate the following integral. (You may use formulas 1-18, 39-42, 50, 51 only from the table of integrals for this problem.)

$$\int \tan^3(2x)\sec^4(2x)\,dx$$

$u = \tan(2x)$ $\quad du = 2\sec^2(2x)dx$. So $\sec^2(2x)\,dx = \frac{1}{2}du$

$\int \tan^3(2x)\cdot\sec^2(2x)\cdot\sec^2(2x)\,dx$

$= \int u^3\sec^2(2x)\cdot\frac{1}{2}du$

$= \frac{1}{2}\int u^3(\tan^2(2x)+1)\,du$

$= \frac{1}{2}\int u^3(u^2+1)\,du$

$= \frac{1}{2}\int (u^5+u^3)\,du$

$= \frac{1}{2}\left[\frac{u^6}{6}+\frac{u^4}{4}\right]$

$= \frac{1}{2}\left[\frac{\tan^6(2x)}{6}+\frac{\tan^4(2x)}{4}\right]+C$.

3. (8 points) Evaluate the following integral. (You may use formulas 1-18, 39-42, 50, 51 only from the table of integrals for this problem.)

$$\int \sqrt{64+x^2}\,dx$$

$x = 8\tan t$

$dx = 8\sec^2 t\,dt$

$$\int\sqrt{64+x^2}\,dx = \int\sqrt{64+64\tan^2 t}\cdot 8\sec^2 t\,dt$$

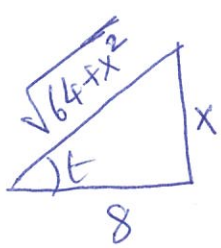
$$= 8\int\sqrt{64(1+\tan^2 t)}\cdot \sec^2 t\,dt$$

$$= 8\int\sqrt{64\sec^2 t}\cdot\sec^2 t\,dt$$

$$= 64\int\sec^3 t\,dt$$

$$= 64\left[\frac{\sec t\tan t}{2} + \frac{1}{2}\int\sec t\,dt\right]$$ Formula 57 with $n=3, a=1$.

$$= 64\left[\frac{\sec t\tan t}{2} + \frac{\ln|\sec t+\tan t|}{2}\right]$$ Formula 17.



$\tan t = \frac{x}{8}$. $\left(\tan t = \frac{\text{opp. side}}{\text{adjacent side}}\right)$.

$\sec t = \frac{\sqrt{64+x^2}}{8}$ $\left(\sec t = \frac{\text{hypotenuse}}{\text{adjacent side}}\right)$.

$$\int\sqrt{64+x^2}\,dx = 32\left[\frac{\sqrt{64+x^2}}{8}\cdot\frac{x}{8} + \ln\left|\frac{\sqrt{64+x^2}}{8}+\frac{x}{8}\right|\right] + C.$$

4. (8 points) Evaluate the following integral. (You may use formulas 1-18 only from the table of integrals for this problem.)

$$\int \frac{11+3x+2x^2}{(x^2+16)(x-7)}\,dx$$

This is a proper rational function since degree of numerator < degree of denominator.

Partial fraction decomposition:

$$\frac{11+3x+2x^2}{(x^2+16)(x-7)} = \frac{Ax+B}{x^2+16} + \frac{C}{x-7}.$$

$$11+3x+2x^2 = (Ax+B)(x-7)+C(x^2+16).$$

$\underline{x=7:}$ $130 = 65C$. So $\boxed{C=2}$

$$11+3x+2x^2 = (Ax+B)(x-7)+2(x^2+16).$$

$\underline{x=0:}$ $11 = B(-7)+32$. So $\boxed{B=3}$.

$$11+3x+2x^2 = (Ax+\cancel{B}^{3})(x-7)+2(x^2+16)$$

$\underline{x=1:}$ $16 = (A+3)(-6)+2(17)$ So $\boxed{A=0}$.

$$\int \frac{11+3x+2x^2}{(x^2+16)(x-7)}\,dx = \int\left(\frac{3}{x^2+16} + \frac{2}{x-7}\right)dx$$

$$= \int \frac{3}{x^2+16}\,dx + \int \frac{2}{x-7}\,dx$$

$u = x-7,\ du = dx.$

$$= 3 \cdot \frac{1}{4}\arctan\left(\frac{x}{4}\right) + \int \frac{2}{u}\,du$$

(Formula 13, $a=4$)

$$= \frac{3}{4}\arctan\left(\frac{x}{4}\right) + 2\ln|u| + C$$

$$= \frac{3}{4}\arctan\left(\frac{x}{4}\right) + 2\ln|x-7| + C.$$

5. (6 points) Use comparison to determine the convergence of the following integral.

$$\int_8^\infty \frac{2\cos x + 3}{x^4+7}\,dx$$

For $x \geq 8$, $-1 \leq \cos x \leq 1$.

$$-2 \leq 2\cos x \leq 2.$$

$$1 \leq 2\cos x + 3 \leq 5$$

$$\frac{1}{x^4+7} \leq \underbrace{\frac{2\cos x+3}{x^4+7} \leq \frac{5}{x^4+7} \leq \frac{5}{x^4}}_{\text{These comparisons are useful here.}} \quad \left[\text{Since } x^4+7 \geq x^4, \ \frac{5}{x^4+7} \leq \frac{5}{x^4}.\right]$$

$$\frac{2\cos x+3}{x^4+7} \leq \frac{5}{x^4}.$$

$\int_8^\infty \frac{5}{x^4}\,dx$ converges. $(p=4>1)$.

So by comparison, $\int_8^\infty \frac{2\cos x+3}{x^4+7}\,dx$ converges.

6. (7 points) The p.d.f. of a continuous random variable X is given by $f(x) = ke^{-0.2x}$ for $x \geq 0$ (the function is zero for all other values of x). Find k. ~~Assume $k>0$~~.

$\int_0^\infty f(x)\,dx = 1$ since $f(x)$ is the p.d.f. of a continuous random variable.

First we find $\int_0^\infty f(x)\,dx$.

$$\int_{\cancel{0}}^{\cancel{\infty}} ke^{-0.2x}\,dx \underset{\substack{u=-0.2x \\ du=-0.2dx}}{=\!=\!=} \int ke^{u}\frac{1}{-0.2}\,du = \frac{-1}{0.2}ke^{u} = \frac{-1}{0.2}ke^{-0.2x}.$$

$$\int_0^t ke^{-0.2x}\,dx = \frac{-1}{0.2}ke^{-0.2x}\Big]_0^t = \frac{-1}{0.2}ke^{-0.2t} + \frac{k}{0.2}$$

$$\lim_{t\to\infty}\int_0^t ke^{-0.2x}\,dx = \lim_{t\to\infty}\left[\frac{-1}{0.2}ke^{-0.2t} + \frac{k}{0.2}\right]$$

$$= \lim_{t\to\infty}\left[\frac{-k}{0.2e^{0.2t}}\right] + \frac{k}{0.2}$$

As $t\to\infty$, $0.2t\to\infty$. So $e^{0.2t}\to\infty$.

Hence $\frac{1}{e^{0.2t}} \to 0$.

So $\frac{-k}{0.2e^{0.2t}} \to 0$.

Thus $\int_0^\infty ke^{-0.2x}\,dx = \lim_{t\to\infty}\int_0^t ke^{-0.2x}\,dx = \frac{k}{0.2}$.

Hence $\frac{k}{0.2} = 1$

ie. $k = 0.2$

7. (7 points) Suppose the fourth-order Taylor polynomial for a function f based at $x_0 = 1$ is

$$P_4(x) = (x-1) + 2(x-1)^2 + 3(x-1)^4 .$$

(a) Write the third-order Taylor polynomial for f based at $x_0 = 1$.

$P_3(x) = (x-1) + 2(x-1)^2$.

(b) What does Taylor's theorem imply about the maximum approximation error committed by P_4 over the interval $[-2, 2]$? Assume that $-4 \le f^{(5)}(x) \le 2$ for all x in $[-2, 2]$.

By Taylor's theorem,

$$|f(x) - P_4(x)| \le \frac{K_5}{5!} |x-1|^5 .$$

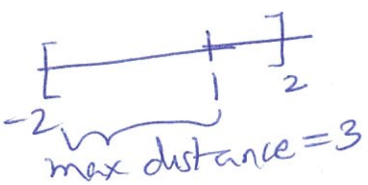
Since $-4 \le f^{(5)}(x) \le 2$ for all x in $[-2,2]$,

$|f^{(5)}(x)| \le 4$ for all x in $[-2,2]$.

So $K_5 = 4$.

$$|f(x) - P_4(x)| \le \frac{4}{5!} |x-1|^5 .$$

For x in $[-2,2]$, max value of $|x-1| = \cancel{2}\, 3$. since the maximum distance from 1 in the interval is 3.



Hence max value of $|x-1|^5 = 3^5$ in $[-2,2]$.

So $|f(x) - P_4(x)| \le \frac{4}{5!} \cdot 3^5 = \cancel{32.4}\ 8.1$

Thus the maximum approximation error committed in the interval $[-2,2]$ is $\cancel{32.4}$ 8.1.