Math 105: Review for Exam II - Solutions

1. **Find dy/dx for each of the following.**

   (a) $\boldsymbol{y = x^2 + 2^x + e^2 + e^{2x} + \ln 2 + \ln(2x) + (\ln 2)x + \arctan 2}$

   $$\frac{dy}{dx} = 2x + (\ln 2)2^x + 2e^{2x} + \frac{1}{2x}\cdot 2 + \ln 2$$

   Note that e^2, $\ln 2$, and $\arctan 2$ are constants.

   (b) $\boldsymbol{y = \sqrt{x}\cdot\arctan(5x)}$

   $$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}\arctan(5x) + \sqrt{x}\cdot\frac{1}{1+(5x)^2}\cdot 5 = \frac{\arctan(5x)}{2x^{1/2}} + \frac{5\sqrt{x}}{1+25x^2}$$

   (c) $\boldsymbol{y = \ln(\tan(2^{\cos(x^2)}))}$

   $$\frac{dy}{dx} = \frac{1}{\tan(2^{\cos(x^2)})}\cdot\sec^2(2^{\cos(x^2)})\cdot\ln 2(2^{\cos(x^2)})\cdot(-\sin(x^2))\cdot 2x$$

   (d) $\boldsymbol{y = \dfrac{x+e^{\pi}}{\cos 4 + \sin^5(6x)}}$

   Note that e^{π} and $\cos 4$ are constants.

   $$\frac{dy}{dx} = \frac{(1)(\cos 4 + \sin^5(6x)) - (x+e^{\pi})(5\sin^4(6x)\cdot\cos(6x)\cdot 6)}{(\cos 4 + \sin^5(6x))^2}$$

   Recall that $\sin^5(6x) = (\sin(6x))^5$.

   (e) $\boldsymbol{y = (x^2+1)^{\sin x}}$

   We need logarithmic differentiation here.

   $$\ln y = \sin x\cdot\ln(x^2+1)$$ Take natural log of each side.

   $$\frac{1}{y}\frac{dy}{dx} = \cos x\cdot\ln(x^2+1) + \sin x\cdot\frac{1}{x^2+1}\cdot 2x$$ Differentiate.

   $$\frac{dy}{dx} = \left[\cos x\cdot\ln(x^2+1) + \frac{2x\sin x}{x^2+1}\right]y$$ Solve for $\frac{dy}{dx}$.

   $$\frac{dy}{dx} = \left[\cos x\cdot\ln(x^2+1) + \frac{2x\sin x}{x^2+1}\right]\cdot(x^2+1)^{\sin x}$$ Replace y.

2. **Consider the curve defined by $\boldsymbol{x^3 + y^3 = \frac{9}{2}xy}$ (known as the Folium of Descartes).**

   (a) **Find dy/dx.**

   Use implicit differentiation.

   $$3x^2 + 3y^2\frac{dy}{dx} = \frac{9}{2}y + \frac{9}{2}x\frac{dy}{dx}$$
   $$3y^2\frac{dy}{dx} - \frac{9}{2}x\frac{dy}{dx} = \frac{9}{2}y - 3x^2$$
   $$\frac{dy}{dx}\left(3y^2 - \frac{9}{2}x\right) = \frac{9}{2}y - 3x^2$$
   $$\frac{dy}{dx} = \frac{\frac{9}{2}y - 3x^2}{3y^2 - \frac{9}{2}x}$$

   (b) **Verify that the point (1,2) is on the curve above.**

   We must check to see if the values $x = 1$ and $y = 2$ satisfy the equation above.

   $$x^3 + y^3 \stackrel{?}{=} \frac{9}{2}xy$$
   $$1^3 + 2^3 \stackrel{?}{=} \frac{9}{2}\cdot 1\cdot 2$$
   $$9 \stackrel{?}{=} 9$$

   Thus, the point (1,2) is on the curve.

(c) **Find the equation of the tangent line at the point (1,2).**

We want $y = mx + b$.

$$m = \frac{\frac{9}{2}\cdot 2 - 3\cdot 1^2}{3\cdot 2^2 - \frac{9}{2}\cdot 1} = \frac{4}{5}, \text{ so } y = \frac{4}{5}x + b.$$

Now plug in $x = 1$ and $y = 2$ to find b.

$$2 = \frac{4}{5}\cdot 1 + b \Rightarrow \frac{6}{5} = b$$

Therefore, we have $y = \frac{4}{5}x + \frac{6}{5}$.

3. **Evaluate the following limits. [Students in the 8:00 and 1:10 sections may omit this problem.]**

Throughout this solution, the symbol ★ will stand for whatever notation your instructor prefers for using L'Hopital's Rule on the indeterminate form 0/0; this may be $\overset{\text{“0/0”}}{=}$ or $\overset{L'H}{=}$ or $\overset{H}{=}$ or = “0/0” or “has the form ‘$\frac{0}{0}$’ and so, by L'Hopital's Rule, is equal to” or something else. The symbol ♡ will serve the same purpose for the indeterminate forms ∞/∞ and $-\infty/\infty$.

(a) $\displaystyle \lim_{x\to 1} \frac{x^3-1}{7-7x} \bigstar \lim_{x\to 1}\frac{3x^2}{-7} = \frac{3}{-7} = -\frac{3}{7}$

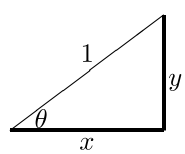
(b) $\displaystyle \lim_{x\to 0} \frac{1-\cos(2x)}{3^x} = \frac{0}{1} = 0$ Can't use (and don't need) L'Hopital's Rule!

(c) $\displaystyle \lim_{x\to 0} \frac{1-\cos(4x)}{5x^2} \bigstar \lim_{x\to 0}\frac{4\sin(4x)}{10x} \bigstar \lim_{x\to 0}\frac{16\cos(4x)}{10} = \frac{16}{10} = \frac{8}{5}$

(d) $\displaystyle \lim_{x\to \infty} \frac{x^2}{2^x} \heartsuit \lim_{x\to\infty}\frac{2x}{\ln 2\cdot 2^x} \heartsuit \lim_{x\to\infty}\frac{2}{\ln 2\cdot \ln 2\cdot 2^x} = 0$

4. **Rewrite $\tan(\arccos x)$ as an algebraic expression - no trigonometric or inverse trigonometric functions. [Students in the 1:10 section may omit this problem.]**

Let $\theta = \arccos x$. That is, θ is the angle whose cosine is x.



$$x^2 + y^2 = 1^2 \Rightarrow y = \sqrt{1-x^2}$$

$$\tan(\arccos x) = \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} = \frac{\sqrt{1-x^2}}{x}$$

5. **Consider the function $f(x) = x^4e^x$ with domain all real numbers.**

(a) **Find the x-value(s) of all roots (x-intercepts) of f.**

The equation $x^4e^x = 0$ means $x^4 = 0$ (that is, $x = 0$) or $e^x = 0$ (no solution), so the only root is at $x = 0$.

(b) **Find the x- and y-value(s) of all critical points and identify each as a local max, local min, or neither.**

$$f'(x) = 4x^3e^x + x^4e^x$$
$$0 = x^3e^x(4+x)$$
$$\Rightarrow x = 0, -4$$

Note that e^x is never 0.

	$x < -4$	$-4 < x < 0$	$4 < x$
f'	positive	negative	positive
f	$\nearrow$	$\searrow$	$\nearrow$

y-values: $f(-4) = 256e^{-4} \approx 4.689$, $f(0) = 0$
So, f has a local maximum at $(-4, 256e^{-4})$ and a local minimum at $(0, 0)$.

(c) **Find the x- and y-value(s) of all global extrema and identify each as a global max or global min.**
There is a global minimum at $(0, 0)$. There is no global maximum because as $x \to \infty$, $f(x) \to \infty$. Note that as $x \to -\infty$, $f(x) \to 0$. You can verify this by using L'Hopital's Rule on x^4/e^{-x}.

(d) **Find the x-value(s) of all inflection points.**

$f''(x) = 12x^2e^x + 4x^3e^x + 4x^3e^x + x^4e^x$ Use Product Rule on each product in $f'(x)$ above.

$0 = e^x(x^4 + 8x^3 + 12x^2)$

$0 = e^x x^2(x^2 + 8x + 12)$

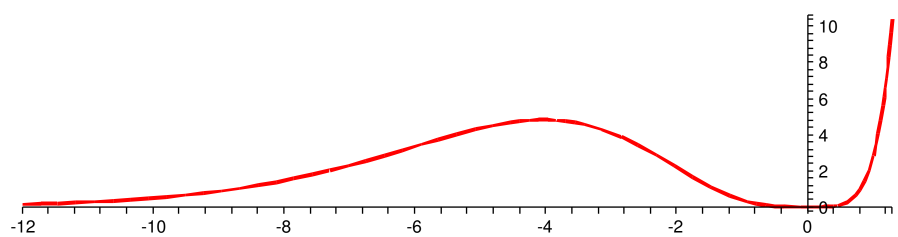
$0 = e^x x^2(x + 2)(x + 6)$

$\Rightarrow x = 0, -2, -6$

	$x < -6$	$-6 < x < -2$	$-2 < x < 0$	$0 < x$
f''	positive	negative	positive	positive
f	concave up	concave down	concave up	concave up

So, the x-values of the inflection points of f are $x = -2$ and $x = -6$ but NOT $x = 0$.

(e) **Sketch f.**



6. **How would your answers to the previous question change if the domain of f were $[-10, 10]$?**
There would be a global maximum at $(10, 10^4e^{10})$. (And the graph would be restricted to $-10 \leq x \leq 10$).

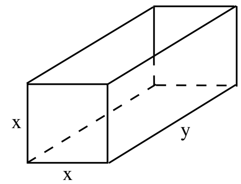
7. **Use the Intermediate Value Theorem to explain why $f(x) = x^3 - 4x^2 + 5$ must have a root somewhere on the interval $[1, 2]$. [Students in the 8:00 and 1:10 sections may omit this problem.]**

IVT: If f is continuous on $[a, b]$ and y is a number between $f(a)$ and $f(b)$, then there is a number c between a and b such that $f(c) = y$.

Our function f is continuous on $[1, 2]$. We can compute that $f(1) = 2$ and $f(2) = -3$. Since 0 is a number between 2 and -3, the IVT says there is a number c between 1 and 2 such that $f(c) = 0$; this c is the desired root.

[In plainer English, f is positive at one endpoint and negative at the other. Since f is continuous, the only way its value can go from positive to negative is to go through zero; where f is zero is our root.]

8. **You are planning to build a box-shaped aquarium with no top and with two square ends. Your budget is \$288. If the glass for the sides costs \$12 per square foot and the opaque material for the bottom costs \$3 per square foot, what dimensions will maximize the volume? Be sure to show how you know you have found the maximum.**



Goal: Maximize volume

Objective function: volume $= V = x \cdot x \cdot y = x^2 y$

We need to get this down to a function of just one variable, so we use the *constraint equation*:

$$\begin{aligned}
\text{total cost} =& (\text{cost of base}) + (\text{cost of two square ends}) + (\text{cost of two other sides}) \\
288 =& \quad 3xy \quad + \quad 12 \cdot 2x^2 \quad + \quad 12 \cdot 2xy \\
288 =& \; 27xy + 24x^2 \\
288 - 24x^2 =& \; 27xy \\
\frac{288 - 24x^2}{27x} =& \; y
\end{aligned}$$

Substituting this back into the objective function gives

$$V = x^2 y = x^2 \cdot \frac{288 - 24x^2}{27x} = x \cdot \frac{288 - 24x^2}{27} = \frac{1}{27}(288x - 24x^3).$$

Now that we have V as a function of just one variable, we find its maximum.

$$\begin{aligned}
V'(x) &= \frac{1}{27}(288 - 72x^2) \\
0 &= \frac{1}{27}(288 - 72x^2) \\
0 &= (288 - 72x^2) \\
72x^2 &= 288 \\
x^2 &= \frac{288}{72} \\
x &= 2 \qquad \text{We discard } x = -2 \text{ because lengths must be nonnegative.}
\end{aligned}$$

Since V' is positive for $x < 2$ and negative for $2 < x$, we know that the maximum occurs at $x = 2$. And $y = \frac{288 - 24x^2}{27x} = \frac{288 - 24 \cdot 2^2}{27 \cdot 2} = \frac{32}{9}$, so the dimensions are 2 by 2 by $\frac{32}{9}$.