Mid-term Exam #2
MATH 106 - A&B Winter 2016

Name: ____

Instructions:

- Answer as many of the following questions as possible.
- No cell phones or collaboration allowed. If you leave the classroom during the exam you must leave your cell phone with the instructor.
- Approved calculators are allowed.
- Additional scrap paper is available upon request.
- *Multiple choice questions:* Circle the letter corresponding to your answer. No partial credit will be awarded.
- *Short answer questions:* Show all of your work on the page of the problem. Clearly indicate your answer and the reasoning that you used to arrive at the answer. You do not have to simplify algebraic expressions.

This exam has 4 multiple choice problems and 5 short answer problems. There are a total of 100 points.

Good luck!

Problem	**Possible Points**	**Points Earned**
MC	20	
5	30	
6	16	
7	20	
8	8	
9	6	
TOTAL	**100**	

1. (5 points) Which of the following is the correct form for the partial fraction decomposition of

$$\frac{x+5}{x^2(x^2+1)}?$$

A. $\dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

B. $\dfrac{A}{x^2} + \dfrac{B}{x^2+1}$

C. $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+1}$

D. $\dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+1}$

E. $\dfrac{Ax+B}{x^2} + \dfrac{Cx+D}{x^2+1}$

SOLUTION: Correct answer: C.

2. (5 points) Which substitution is needed to evaluate the integral

$$\int \tan^4 x \sec^2 x \, dx?$$

A. $u = \cos x$, $du = -\sin x \, dx$

B. $u = \sec x$, $du = \tan x \sec x \, dx$

C. $u = 1 - \tan^2 x$, $du = 2\tan x \sec^2 x \, dx$

D. $u = \tan x$, $du = \sec^2 x \, dx$

E. $u = \tan^4 x$, $du = \sec^2 x \, dx$

SOLUTION: Correct answer: D.

3. (5 points) Suppose that the third-order Maclaurin polynomial for a function f is given by

$$P_3(x) = 4 + \frac{x}{6} - \frac{2x^2}{3} - 12x^3.$$

What is $f''(0)$?

A. $f''(0) = -\frac{4}{3}$

B. $f''(0) = -\frac{2}{3}$

C. $f''(0) = \frac{1}{6}$

D. $f''(0) = \frac{1}{2}$

E. $f''(0) = 0$

SOLUTION: Correct answer: A.

4. (5 points) The improper integral $\displaystyle\int_{-\infty}^{\infty} x\,dx$ is

A. convergent since the area to the left of $x = 0$ cancels with the area to the right of $x = 0$.

B. convergent since it equals $\displaystyle\lim_{t\to-\infty}\int_t^0 x\,dx + \lim_{t\to\infty}\int_0^t x\,dx = -\infty + \infty = 0.$

C. divergent by comparison to $\displaystyle\int_{-\infty}^{\infty} xe^{-x}\,dx$

D. divergent since $\displaystyle\int_{-\infty}^{0} x\,dx$ is divergent and $\displaystyle\int_0^{\infty} x\,dx$ is convergent.

E. divergent since both integrals $\displaystyle\int_{-\infty}^{0} x\,dx$ and $\displaystyle\int_0^{\infty} x\,dx$ are divergent.

SOLUTION: E.

5. (30 points) Evaluate the following integrals using calculus. Show your work.

(a) (10 points) $\displaystyle\int \frac{5x+8}{x^2+2x-8}\,dx$

SOLUTION: By partial fraction decomposition,

$$\frac{5x+8}{x^2+2x-8} = \frac{A}{x+4} + \frac{B}{x-2}.$$

Solving for A, B we get $A = 2$ and $B = 3$. Then

$$\begin{aligned} \int \frac{5x+8}{x^2+2x-8}\,dx &= \int \frac{2}{x+4}\,dx + \int \frac{3}{x-2}\,dx \\ &= 2\ln|x+4| + 3\ln|x-2| + C. \end{aligned}$$

(b) (10 points) $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\,dx$

SOLUTION: Use a trig substitution. Let $x = \sec\theta$, then $dx = \sec\theta\tan\theta\,d\theta$.

$$\begin{aligned} \int \frac{\sqrt{x^2-1}}{x}\,dx &= \int \frac{\sqrt{\sec^2\theta-1}}{\sec\theta}\sec\theta\tan\theta\,d\theta \\ &= \int \sqrt{\tan^2\theta}\tan\theta\,d\theta \\ &= \int \tan^2\theta\,d\theta \\ &= \int (\sec^2\theta - 1)\,d\theta \\ &= \tan\theta - \theta + C. \end{aligned}$$

Build a right triangle with the identity $\sec\theta = x$, or equivalently $\cos\theta = \frac{1}{x}$. Then the adjacent side has length 1, the hypotenuse has length x, and the opposite side has length $\sqrt{x^2-1}$. Thus $\tan\theta = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$, and

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \sqrt{x^2-1} - \arccos\left(\frac{1}{x}\right) + C.$$

(c) (10 points) Compute ONE of the following indefinite integrals. *You will not receive extra credit for attempting both problems. If you attempt both problems make it clear which problem that you would like graded.*

i. $\int \arcsin(x)\, dx$

ii. $\int \sin(\ln x)\, dx$

SOLUTION: (a) Use integration by parts, with $u = \arcsin x$ and $dv = dx$. Then $du = \frac{1}{\sqrt{1-x^2}}\, dx$ and $v = x$. We have,

$$\int \arcsin(x)\, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\, dx.$$

Now let $w = 1 - x^2$, then $dw = -2x\, dx$. Thus by substitution,

$$\begin{aligned}
\int \frac{x}{\sqrt{1-x^2}}\, dx &= -\frac{1}{2} \int \frac{dw}{\sqrt{w}} \\
&= -\frac{1}{2}\left(2\sqrt{w}\right) + C \\
&= -\sqrt{w} + C \\
&= -\sqrt{1-x^2} + C.
\end{aligned}$$

Combining this with the result above, we have

$$\int \arcsin(x)\, dx = x \arcsin x + \sqrt{1-x^2} + C.$$

(b) For (b), again use integration by parts. Let $u = \sin(\ln x)$ and $dv = dx$. Then $du = \frac{\cos(\ln x)}{x}\, dx$ and $v = x$.

$$\begin{aligned}
\int \sin(\ln x)\, dx &= x \sin(\ln x) - \int \frac{\cos(\ln x)}{x} x\, dx \\
&= x \sin(\ln x) - \int \cos(\ln x)\, dx.
\end{aligned}$$

Integrate by parts again, this time with $u = \cos(\ln x)$, $dv = dx$, and $du = \frac{-\sin(\ln x)}{x}\, dx$ and $v = x$. Thus,

$$\begin{aligned}
\int \sin(\ln x)\, dx &= x \sin(\ln x) - \left(x \cos(\ln x) - \int \frac{-\sin(\ln x)}{x} x\, dx \right) \\
&= x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x)\, dx.
\end{aligned}$$

Finally, solving for $\int \sin(\ln x)\,dx$ we get

$$\begin{aligned} 2\int \sin(\ln x)\,dx &= x\sin(\ln x) - x\cos(\ln x) \\ \int \sin(\ln x)\,dx &= \frac{x\sin(\ln x) - x\cos(\ln x)}{2} + C. \end{aligned}$$

6. (16 points) Determine if the following integrals are convergent or divergent. If convergent, evaluate the integral using calculus. Show your work.

(a) (8 points) $\displaystyle\int_1^9 \frac{1}{(x-4)^2}\,dx$

SOLUTION: The integrand is infinite at $x = 4$, so we break the integral at this point and look at

$$\begin{aligned}
\int_1^9 \frac{dx}{(x-4)^2} &= \int_1^4 \frac{dx}{(x-4)^2} + \int_4^9 \frac{dx}{(x-2)^2} \\
&= \lim_{t\to 4^-} \int_1^t \frac{dx}{(x-4)^2} + \lim_{t\to 4^+} \int_t^9 \frac{dx}{(x-4)^2} \\
&= \lim_{t\to 4^-} \left.\frac{-1}{x-4}\right|_1^t + \lim_{t\to 4^+} \left.\frac{-1}{x-4}\right|_t^9 \\
&= \lim_{t\to 4^-} \left(\frac{-1}{t-4} - \frac{1}{3}\right) + \lim_{t\to 4^+} \left(\frac{-1}{5} + \frac{1}{t-4}\right).
\end{aligned}$$

Both limits diverge, so the improper integral diverges.

(b) (8 points) $\displaystyle\int_{-\infty}^{-2} \frac{dx}{1-x}$

SOLUTION: This integral is improper because the interval is infinite. Replace the $-\infty$ with t and take a limit as t approaches $-\infty$:

$$\begin{aligned}
\int_{-\infty}^{-2} \frac{dx}{1-x} &= \lim_{t\to-\infty} \int_t^{-2} \frac{dx}{1-x} \\
&= \lim_{t\to-\infty} \left. -\ln(1-x)\right|_t^{-2} \\
&= \lim_{t\to-\infty} \left(-\ln 3 + \ln(1-t)\right)
\end{aligned}$$

The limit above diverges, so the improper integral diverges.

7. (20 points) For this problem, consider $f(x) = \ln(x^2)$.

(a) (10 points) Find the fourth-order Taylor polynomial $P_4(x)$ for $f(x)$ centered at $x_0 = 1$. *You do not have to simplify coefficients.*

SOLUTION: First, compute the first four derivatives of f and evaluate them at the point $x_0 = 1$:

$f(x) = \ln(x^2)$	$f(1) = 0$
$f'(x) = \frac{2x}{x^2} = \frac{2}{x}$	$f'(1) = 2$
$f''(x) = \frac{-2}{x^2}$	$f''(1) = -2$
$f'''(x) = \frac{4}{x^3}$	$f'''(1) = 4$
$f^{(4)}(x) = \frac{-12}{x^4}$	$f^{(4)}(1) = -12$

Then

$$\begin{aligned} P_3(x) &= 0 + 2(x-1) - \frac{2}{2}(x-1)^2 + \frac{4}{3!}(x-1)^3 - \frac{12}{4!}(x-1)^4 \\ &= 2(x-1) - (x-1)^2 + \frac{2}{3}(x-1)^3 - \frac{1}{2}(x-1)^4. \end{aligned}$$

(b) (10 points) Taylor's Theorem states that $|f(x) - P_n(x)| \le \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}$ for all values of x in an interval I containing x_0. What is the maximum error committed by using $P_4(x)$ (as in part (a)) over the interval $[\frac{1}{2}, \frac{3}{2}]$? *You do not have to simplify your answer.*

SOLUTION: For this problem, $n = 4$. We must find K_5 and an upper bound for $|x - 1|^5$ on the interval $[\frac{1}{2}, \frac{3}{2}]$.

First, to find K_5 I seek an upper bound for $|f^{(5)}(x)|$ over the interval $[\frac{1}{2}, \frac{3}{2}]$. We have $f^{(5)}(x) = \frac{48}{x^5}$. On the interval $[\frac{1}{2}, \frac{3}{2}]$, this function is always positive and it is largest at $x = \frac{1}{2}$. Thus

$$|f^{(5)}(x)| \le \frac{48}{(1/2)^5} = 1536.$$

I will use $K_5 = 1536$.

Next, for x in the interval $[\frac{1}{2}, \frac{3}{2}]$, $|x - 1| \le \frac{1}{2}$, so $|x - 1|^5 \le \frac{1}{2^5} = \frac{1}{32}$.

Then by Taylor's theorem,

$$|f(x) - P_4(x)| \le \frac{1536}{4!}\frac{1}{32} = 2.$$

8. (8 points) Use comparison to determine if the following improper integral converges or diverges. If the integral converges, find an upper bound for its value. Show all of your work to justify your answer.

$$\int_1^\infty \frac{x^2+1}{x^4+x}\,dx$$

SOLUTION: For $x \geq 1$, $x^4 + x \geq x^4$. Then $\dfrac{1}{x^4+x} \leq \dfrac{1}{x^4}$, and moreover $\dfrac{x^2+1}{x^4+x} \leq \dfrac{x^2+1}{x^4}$. Then

$$\int_1^\infty \frac{x^2+1}{x^4+x}\,dx \leq \int_1^\infty \frac{x^2+1}{x^4}\,dx = \int_1^\infty \frac{x^2}{x^4} + \frac{1}{x^4}\,dx.$$

Split the last integral above into two separate integrals. In the first integral, we see that

$$\int_1^\infty \frac{x^2}{x^4}\,dx = \int_1^\infty \frac{1}{x^2}\,dx.$$

This integral converges by the p-test to 1.

The second integral is $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx$, and this integral again converges by the p test to $\dfrac{1}{3}$.

Then $\displaystyle\int_1^\infty \frac{x^2+1}{x^4+x}\,dx$ converges, and moreover $\displaystyle\int_1^\infty \frac{x^2+1}{x^4+x}\,dx \leq 1 + \frac{1}{3} = \frac{4}{3}$.

9. (6 points) Suppose that the probability density function of a random variable X is as follows:

$$f(x) = \begin{cases} ce^{-x/3} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) (4 points) Find the value of the constant c.

SOLUTION: We need $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Beginning with the left-hand side,

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} ce^{-x/3}\,dx \\ &= \lim_{t\to\infty} \int_{0}^{t} ce^{-x/3}\,dx \\ &= c \lim_{t\to\infty} -3e^{-x/3}\Big|_0^t \\ &= -3c \lim_{t\to\infty} \left(e^{-t/3} + 1\right) \\ &= -3c(1) = -3c. \end{aligned}$$

Now set $-3c = 1$ and solve for c:

$$c = -\frac{1}{3}.$$

(b) (2 points) Find the probability that $X \leq \frac{1}{4}$.

SOLUTION: The probability that $X \leq \frac{1}{4}$ is given by

$$\begin{aligned} \int_{-\infty}^{1/4} f(x)\,dx &= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1/4} -\frac{1}{3}e^{-x/3}\,dx \\ &= -\frac{1}{3}\left(-3e^{-x/3}\Big|_0^{1/4}\right) \\ &= e^{-1/12} + 1 \\ &= 1 - e^{-1/12} \approx 0.07996. \end{aligned}$$

Note that the integral above is not improper.