

Math 106: Review for Exam II

INTEGRATION TIPS

- Substitution: usually let $u =$ a function that's "inside" another function, especially if du (possibly off by a multiplying constant) is also present in the integrand.

- Parts: $\int u dv = uv - \int v du$ or $\int uv' dx = uv - \int u'v dx$

How to choose which part is u ? Let u be the part that is higher up in the **LIATE** mnemonic below. (The mnemonics **ILATE** and **LIPET** will work equally well if you have learned one of those instead; in the latter **A** is replaced by **P**, which stands for "polynomial.")

Logarithms (such as $\ln x$)

Inverse trig (such as $\arctan x, \arcsin x$)

Algebraic (such as $x, x^2, x^3 + 4$)

Trig (such as $\sin x, \cos 2x$)

Exponentials (such as e^x, e^{3x})

- Rational Functions (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

Partial Fractions: here's an illustrative example of the setup.

$$\frac{3x^2 + 11}{(x + 1)(x - 3)^2(x^2 + 5)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{Dx + E}{x^2 + 5}$$

Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor $(x - 3)$ on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term ($Dx + E$ here) above it on the right.

- Trigonometric Substitutions: some suggested substitutions and useful formulae follow.

| Radical Form | $\sqrt{a^2 - x^2}$ | $\sqrt{a^2 + x^2}$ | $\sqrt{x^2 - a^2}$ |
|--------------|--------------------|--------------------|--------------------|
| Substitution | $x = a \sin t$ | $x = a \tan t$ | $x = a \sec t$ |

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 & \tan^2 x + 1 &= \sec^2 x \\ \sin^2 x &= \frac{1}{2} - \frac{\cos(2x)}{2} & \cos^2 x &= \frac{1}{2} + \frac{\cos(2x)}{2} \\ \sin(2x) &= 2 \sin x \cos x \end{aligned}$$

- Powers of Trigonometric Functions: here are some strategies for dealing with these.

| $\int \sin^m x \cos^n x dx$ | Possible Strategy | Identity to Use |
|-----------------------------|--|--|
| m odd | Break off one factor of $\sin x$ and substitute $u = \cos x$. | $\sin^2 x = 1 - \cos^2 x$ |
| n odd | Break off one factor of $\cos x$ and substitute $u = \sin x$. | $\cos^2 x = 1 - \sin^2 x$ |
| m even AND n even | Use $\sin^2 x + \cos^2 x = 1$ to reduce to only powers of $\sin x$ or only powers of $\cos x$, then use table of integrals #39-42 or identities shown to right of this box. | $\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2}$ $\cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2}$ |

| $\int \tan^m x \sec^n x dx$ | Possible Strategy | Identity to Use |
|-----------------------------|---|---------------------------|
| m odd | Break off one factor of $\sec x \tan x$ and substitute $u = \sec x$. | $\tan^2 x = \sec^2 x - 1$ |
| n even | Break off one factor of $\sec^2 x$ and substitute $u = \tan x$. | $\sec^2 x = \tan^2 x + 1$ |
| m even AND n odd | Use identity at right to reduce to powers of $\sec x$ alone. Then use table of integrals #51 or integration by parts. | $\tan^2 x = \sec^2 x - 1$ |

Useful Trigonometric Derivatives and Antiderivatives

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\int \sec x dx = \ln |\sec x + \tan x| + C$$

- Improper integrals: look for ∞ as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration. It may be useful to know the following limits.

$$\lim_{x \rightarrow \infty} e^x =$$

$$\lim_{x \rightarrow \infty} e^{-x} =$$

$$\lim_{x \rightarrow \infty} 1/x =$$

$$\lim_{x \rightarrow 0^+} 1/x =$$

$$\lim_{x \rightarrow \infty} \ln x =$$

$$\lim_{x \rightarrow 0^+} \ln x =$$

Note: this is the same as $\lim_{x \rightarrow -\infty} e^x$

Note: the answer is the same for $\lim_{x \rightarrow \infty} 1/x^2$ and similar functions

Note: the answer is the same for $\lim_{x \rightarrow 0^+} 1/x^2$ and similar functions

1. Evaluate the following.

(a) $\int \sin^6 x \cos^3 x dx$

(b) $\int \frac{dx}{\sqrt{100 + x^2}}$

$$(c) \int_3^{\infty} \frac{1}{x(\ln x)^{100}} dx$$

$$(d) \int_0^{\infty} xe^{-2x} dx$$

$$(e) \int \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} dx$$

$$(f) \int \frac{4x^3 - 27x^2 + 20x - 17}{x - 6} dx$$

$$(g) \int_{-1}^5 \frac{1}{(x - 1)^6} dx$$

2. Find the second-order Taylor polynomial for $f(x) = \sqrt{x}$ based at $x_0 = 100$. Then use your polynomial to estimate $\sqrt{105}$.

3. What is the largest possible error that could have occurred in your estimate of $\sqrt{105}$?

4. Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.

(a)
$$\int_1^{\infty} \frac{6 + \cos x}{x^{0.99}} dx$$

(b)
$$\int_1^{\infty} \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} dx$$

5. The probability density function (pdf) of the length (in minutes) of phone calls on a wireless network is given by $f(x) = ke^{-0.2x}$ where x is the number of minutes. Note that the domain is $x \geq 0$ since we can't have a negative number of minutes.

(a) What must be the value of k ?

(b) What fraction of calls last more than 3 minutes?