1. Use integration by parts to find $\int 3x^2 \arctan(x)\, dx$. Show all your work. (Hint: this problem may involve division of polynomials later on).

"LIATE" recommends trying $u = \arctan(x)$.

Thus, $v' = 3x^2$. So $u' = \frac{1}{1+x^2}$ and $v = x^3$.

The parts formula is $\int uv'\,dx = uv - \int vu'\,dx$

so $\int 3x^2 \arctan(x)\,dx = \arctan(x)\cdot x^3 - \int x^3 \cdot \frac{1}{1+x^2}\,dx$

now, $\frac{x^3}{x^2+1}$ is "improper" meaning we have to carry out the indicated division: $x^2+1 \overline{)\,x^3}$

we find
$$\begin{array}{r|l} & x \\ \hline x^2+1 & x^3 \\ & -(x^3+x) \\ \hline & -x \end{array}$$
quotient

remainder

that is, $\frac{x^3}{x^2+1}$ is $x + \frac{-x}{x^2+1} = x - \frac{x}{x^2+1}$

so $\int \frac{x^3}{x^2+1}\,dx = \int\left(x - \frac{x}{x^2+1}\right)dx = \frac{x^2}{2} - \int \frac{x}{x^2+1}\,dx$.

this last integral is an easy substitution problem:

let $w = x^2+1$, then $dw = 2x\,dx$

so $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x\,dx}{x^2+1} = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w|$

$= \frac{1}{2}\ln|x^2+1|$

(leaving off "+C" til the end)

Finally:

$$\int 3x^2 \arctan(x)\,dx = \boxed{\arctan(x)\cdot x^3 - \left(\frac{x^2}{2} - \frac{1}{2}\ln|x^2+1|\right) + C}$$

2. Show how the substitution $x = 2\cos t$ can be used to find $\int x^3\left(\sqrt{4-x^2}\right)^5\,dx$. (Same as $\int x^3(4-x^2)^{5/2}\,dx$). Show all your work and any relevant triangles.

let $x = 2\cos t$

so $dx = \boxed{-2\sin t\,dt.}$

Also: $x^3 = (2\cos t)^3 = \boxed{8\cos^3 t}$

and $x^2 = 4\cos^2 t$

so $\sqrt{4-x^2} = \sqrt{4-4\cos^2 t}$ ← [NOTE WELL: THIS IS NOT OK to replace with "$2-2\cos t$"; $\sqrt{A^2-B^2} \neq \sqrt{A^2}-\sqrt{B^2}$]

$= \sqrt{4\sin^2 t}$

$= 2\sin t$

and $\left(\sqrt{4-x^2}\right)^5 = (2\sin t)^5 = \boxed{32\sin^5 t}$

Putting all the boxed parts together we get

$$\int x^3\left(\sqrt{4-x^2}\right)^5 dx = \int (8\cos^3 t)(32\sin^5 t)(-2\sin t)\,dt$$

$$= -512\int \cos^3 t\,\sin^6 t\,dt.$$

this is one of those "$\int \cos^n t\,\sin^m t\,dt$" type $\int$'s.

an odd # of cosine factors suggests we keep one for du and turn the others into sines:

$$= -512\int (\cos t)(\cos^2 t)(\sin^6 t)\,dt$$

$$= -512\int (\cos t)(1-\sin^2 t)(\sin^6 t)\,dt \quad \text{Let } u = \sin t;\ du = \cos t\,dt;\ \text{so:}$$

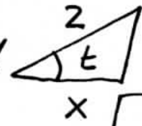
(move it)

$$= -512\int (1-\sin^2 t)(\sin^6 t)(\cos t)\,dt \quad \text{becomes}$$

$$= -512\int (1-u^2)\,u^6\,du$$

$$= -512\int (u^6 - u^8)\,du = -512\left(\frac{u^7}{7} - \frac{u^9}{9}\right) + C$$

$$= -512\left(\frac{\sin^7 t}{7} - \frac{\sin^9 t}{9}\right) + C$$

with $\cos t = x/2 = \frac{\text{adj}}{\text{hyp}}$, the $\triangle$ is labeled 

"and we find the opp. side is $\sqrt{4-x^2}$

Since $\sin t = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{4-x^2}}{2}$ we get, finally:

$$\boxed{-512\left[\frac{\left(\frac{\sqrt{4-x^2}}{2}\right)^7}{7} - \frac{\left(\frac{\sqrt{4-x^2}}{2}\right)^9}{9}\right] + C}$$

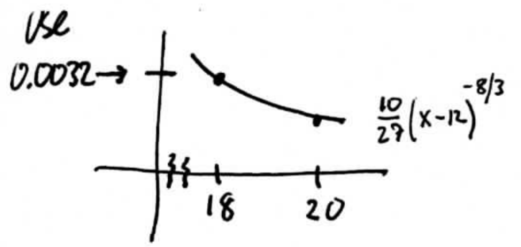
3A. Find $P_2(x)$, the second-order Taylor polynomial approximation of $f(x) = \sqrt[3]{x-12}$, in powers of $x - 20$. Show all your work neatly organized in a table as we've done in class. Constants in your polynomial must be written as fractions (not their decimal equivalents)

K	$f^{(K)}(x)$	$f^{(K)}(a)$	$C_K = \frac{f^{(K)}(a)}{K!}$
0	$f(x) = (x-12)^{1/3}$	$f(20) = (20-12)^{1/3} = 8^{1/3} = 2$	$2/0! = 2/1 = 2$
1	$f'(x) = \frac{1}{3}(x-12)^{-2/3}$	$f'(20) = \frac{1}{3} \cdot 8^{-2/3} = \frac{1}{3} 2^{-2} = \frac{1}{12}$	$(1/12)/1! = 1/12$
2	$f''(x) = \frac{-2}{9}(x-12)^{-5/3}$	$f''(20) = -\frac{2}{9} \cdot 8^{-5/3} = \frac{-2}{9} 2^{-5} = \frac{-2}{9} \cdot \frac{1}{32} = \frac{-1}{144}$	$(-1/144)/2! = -1/288$
3	$f'''(x) = \frac{10}{27}(x-12)^{-8/3}$		

so $\boxed{P_2(x) = 2 + \frac{1}{12}(x-20) - \frac{1}{288}(x-20)^2}$

3B. You can use P_2 to find an approximation of $\sqrt[3]{6}$ by evaluating $P_2(18)$, since $P_2(18) \approx f(18) = \sqrt[3]{18-12} = \sqrt[3]{6}$. Find the maximum possible error in using $P_2(18)$ to approximate $f(18)$ as guaranteed by Taylor's theorem. In your work, choose K_3 correct to 4 decimal places.

a combination of inspecting a table of values of $f'''(x)$ for $x \in [18, 20]$ and a graph of $f'''(x)$ on $[18, 20]$ suggests $K_3 = 0.0032$ will suffice in the error-guarantee formula. we get



$$\frac{K_3 \, |18-20|^3}{3!} = \frac{(0.0032) 2^3}{6} = \boxed{0.0042\overline{6}}$$

3C. Find $P_2(18)$ and also the exact error if $P_2(18)$ is indeed used to approximate $f(18)$.

$P_2(18) = 1.81944...$

$f(18) = \sqrt[3]{6} = 1.81712...$

exact error = the abs. value of their difference; i.e. $\boxed{\approx 0.0023}$ (which is indeed < the guarantee from 3B)

4. Use the method of partial fractions to find $\displaystyle\int \frac{4x^2+27x+71}{(x-1)(x^2+8x+25)}\,dx$

1) Since the deg of the numerator (=2) is LESS than the degree of the denominator (=3) there's no division of polys. required.

2) next, $x^2+8x+25$ doesn't factor. So

the Partial Fraction Decomposition (PFD) has the form $\dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+8x+25}$

After introducing the common denominator, we compare numerators: $4x^2+27x+71 = A(x^2+8x+25)+(Bx+C)(x-1)$

some "easy" x's to use here are $x=1$, $x=0$, $x=2$; in any case you'll find

$A=3$, $C=4$, $B=1$ so

$$\int \frac{4x^2+27x+71}{(x-1)(x^2+8x+25)}\,dx = \int \frac{3}{x-1}\,dx + \int \frac{x+4}{x^2+8x+25}\,dx$$

Try $u=x^2+8x+25$ as a substitution in here; get $du = 2x+8\,dx$, and so the second integral can be rewritten as $\frac{1}{2}\int \frac{2x+8}{x^2+8x+25}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+8x+25| = \frac{1}{2}\ln(x^2+8x+25)$ since u is always positive.

FINAL ANSWER:

$$\boxed{3\ln|x-1| + \frac{1}{2}\ln(x^2+8x+25)+K.}$$

no arctans were needed/used in the solution!

5. Explain why $\displaystyle\int_2^5 \frac{2x}{\sqrt{x^2-4}}\,dx$ is an improper integral. Then determine if it diverges or converges, and if it does converge, find out to what. Use good notation throughout.

As $x\to 2^+$, $\sqrt{x^2-4}\to 0$ so $\dfrac{2x}{\sqrt{x^2-4}}$ has a vertical asymptote at 2.

We agreed to write $\displaystyle\int_2^5 \frac{2x}{\sqrt{x^2-4}}\,dx = \lim_{B\to 2^+}\int_B^5 \frac{2x}{\sqrt{x^2-4}}\,dx$ (provided this limit exists)

now we need an anti-D of $\dfrac{2x}{\sqrt{x^2-4}}$. Let $u=x^2-4$, then $du = 2x\,dx$ and we get

$$\int \frac{2x}{\sqrt{x^2-4}}\,dx = \int \frac{du}{u^{1/2}} = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} = 2\sqrt{u} = 2\sqrt{x^2-4}$$

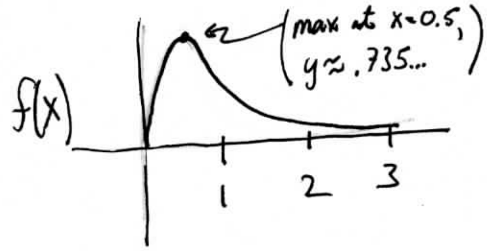
so we can write $\displaystyle\lim_{B\to 2^+}\int_B^5 \frac{2x}{\sqrt{x^2-4}}\,dx = \lim_{B\to 2^+}\left(2\sqrt{x^2-4}\,\Big|_B^5\right)$

$$= \lim_{B\to 2^+}\left(2\sqrt{5^2-4} - 2\sqrt{B^2-4}\right) = (2\sqrt{21} - 2\sqrt{0}) = 2\sqrt{21}$$; this integral CONVERGES

NOTE WELL: if you want to do the substitution and keep the limits on the "$\int$"-sign you must write: $\displaystyle\int_B^5 \frac{2x}{\sqrt{x^2-4}}\,dx = \int_{B^2-4}^{21} \frac{du}{u^{1/2}}$ so that EVERY thing is in terms of $u=x^2-4$!!!

in fact you don't even need to "change back to x's": $= 2u^{1/2}\Big|_{B^2-4}^{21} = 2\sqrt{21} - 2\sqrt{B^2-4}$ directly.

6A. Let $f(x) = 4\,x\,e^{-2x}$. Explain why f is a probability density function on $[0, \infty)$. You can use the fact that an anti-derivative of $f(x)$ is $-(1+2x)e^{-2x}$. Support any claims about limits with a brief table.



In order to be a probability density function on $[0,\infty)$ we need to check that:

① $f(x) \geq 0$ on $[0,\infty)$ but this is true since $4x \geq 0$ and $e^{-2x} \geq 0$...

② $\int_0^\infty f(x)\,dx = 1.$

To show ②: $\int_0^\infty f(x)\,dx = \lim_{B\to\infty} \int_0^B f(x)\,dx = \lim_{B\to\infty} \int_0^B 4xe^{-2x}\,dx = \lim_{B\to\infty} \left(-(1+2x)e^{-2x}\right)\Big|_0^B$

$= \lim_{B\to\infty}\Big(\underbrace{-(1+2B)e^{-2B}}_{\text{this} \to 0 \text{ as shown in the table}} \underbrace{- -(1+2\cdot 0)e^{-2\cdot 0}}_{\text{this is } +1}\Big) = 0 + 1 = 1$ as required

B	$-(1+2B)e^{-2B}$
1	$-.406$
5	-0.000499
10	-0.00000004328
$\downarrow$	$\downarrow$
∞	0 (VERY QUICKLY)

6B. Suppose $f(x)$ represents the distribution of how many decades years LED lightbulbs last before failure. What integral represents the probability that a lightbulb chosen at random will last between 10 and 15 years? (That is, between 1 and 1.5 decades)? Find this probability; show your work.

the integral sought is $\int_1^{1.5} 4x\,e^{-2x}\,dx$

$= -(1+2x)e^{-2x}\Big|_1^{1.5} = -0.199148\ldots - (-0.406005\ldots)$

$= -0.199148\ldots + 0.406005$

$= \boxed{.206857\ldots}$

(same as $-(1+2\cdot 1.5)e^{-2\cdot 1.5} - -(1+2\cdot 1)e^{-2\cdot 1}$
$= -(4)e^{-3} + (3)e^{-2}$)

6C. What is the probability that a bulb will last more than 15 years?

that's the probability that it will FAIL in $[1.5, \infty) = \int_{1.5}^{\infty} 4xe^{-2x}\,dx = -0.199148$ (because it's $= \lim_{B\to\infty} -(1+2x)e^{-2x}\Big|_{1.5}^{B} = \lim_{B\to\infty}\big(\underbrace{-1+2Be^{-2B}}_{\to 0}\big) - \big(\underbrace{-(1+2\cdot 1.5)e^{-2\cdot 1.5}}_{(-4)e^{-3}}\big) = 4e^{-3}$ $\boxed{= .199148}$)