Math 106 Winter 2013

Test 2 (50 points)

Name: ______________________________

Show all your work to receive full credit for a problem.

Do not use the calculator integral function.

When you use a formula from the table of integrals, mention the formula number and the value(s) of any constant(s) that you may need.

Write exact answers. If necessary, round off your answers to four decimal places.

Include units in your answers wherever possible.

There are nine questions. Questions are printed on both sides of a page.

You may use any of the following facts:

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$|f(x) - P_n(x)| \le \frac{K_{n+1}}{(n+1)!}\,|x - x_0|^{n+1}$$

$$\int u\,dv = uv - \int v\,du$$

$\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$ and diverges for $p \le 1$.

$\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converges for $p < 1$ and diverges for $p \ge 1$.

$\displaystyle\int_0^\infty e^{-ax}\,dx$ converges for $a > 0$.

1. (7 points) Evaluate the following integral exactly. (You may use formulas 1-18 only from the table of integrals for this problem.)

$$\int \frac{x^3}{\sqrt{4+x^2}}\, dx$$

$x = 2\tan t$

$dx = 2\sec^2 t\, dt.$

$$\int \frac{x^3}{\sqrt{4+x^2}}\, dx = \int \frac{8\tan^3 t}{\sqrt{4+4\tan^2 t}} \cdot 2\sec^2 t\, dt$$

$$= \int \frac{16\tan^3 t \sec^2 t\, dt}{\sqrt{4(1+\tan^2 t)}}$$

$$= \int \frac{16\tan^3 t \sec^2 t\, dt}{2\sec t}$$

$$= 8\int \tan^3 t \sec t\, dt$$

$$= 8\int \tan^2 t \cdot \tan t \sec t\, dt$$

$$= 8\int (\sec^2 t - 1)\tan t \sec t\, dt$$

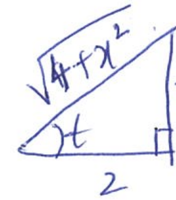
$u = \sec t$

$du = \sec t \tan t\, dt$

$$= 8\int (u^2-1)\, du$$

$$= 8\left[\frac{u^3}{3} - u\right] + C.$$

$$= 8\left[\frac{\sec^3 t}{3} - \sec t\right] + C.$$



$x = 2\tan t$

$\tan t = \frac{x}{2}.$

$\sec t = \frac{\sqrt{4+x^2}}{2}.$

$$= \frac{8}{3}\left(\frac{\sqrt{4+x^2}}{2}\right)^{3} - 8\,\frac{\sqrt{4+x^2}}{2} + C$$

$$= \frac{(\sqrt{4+x^2})^3}{3} - 4\sqrt{4+x^2} + C. \quad \cancel{-\frac{11}{3}\sqrt{4+x^2}+C}$$

2. (7 points) Evaluate the following integral exactly. (You may use formulas 1-18, 39-42, 50, 51 only from the table of integrals for this problem.)

$$\int \frac{11 - 4x + 5x^2}{(x-1)(x^2+5)}\, dx$$

Partial fractions:

$$\frac{11-4x+5x^2}{(x-1)(x^2+5)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+5}.$$

$$11-4x+5x^2 = A(x^2+5) + (Bx+C)(x-1).$$

$\underline{x=1:}$ $12 = 6A.$ $\underline{A=2.}$

$\underline{x=0:}$ $11 = 10 - C.$ $\underline{C=-1.}$

$\underline{x=2:}$ $23 = 17+2B.$ $\underline{B=3.}$

So $$\int \frac{11-4x+5x^2}{(x-1)(x^2+5)}dx = \int \frac{2}{x-1}\,dx + \int \frac{3x-1}{x^2+5}\,dx.$$

$$= \int \frac{2}{x-1}\,dx + \int \frac{3x}{x^2+5}\,dx - \int \frac{1}{x^2+5}\,dx.$$

$u = x-1$ $\quad$ $w = x^2+5$

$du = dx.$ $\quad$ $dw = 2x\,dx.$

$$= \int \frac{2}{u}\,du + \frac{3}{2}\int \frac{dw}{w} - \int \frac{1}{x^2+5}\,dx$$

$$= 2\ln|u| + \frac{3}{2}\ln|w| - \frac{1}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) + C$$

$$= 2\ln|x-1| + \frac{3}{2}\ln|x^2+5| - \frac{1}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) + C.$$

3. (4 points) Evaluate the following integral exactly. (You may use formulas 1-18, 39-42, 50, 51 only from the table of integrals for this problem.)

$$\int \sin^5 x \cos^3 x \, dx$$

$u = \sin x$. $du = \cos x \, dx$.

$$\int \sin^5 x \cos^3 x \, dx = \int \sin^5 x (1 - \sin^2 x) \cos x \, dx$$

$$= \int u^5 (1 - u^2) \, du$$

$$= \int (u^5 - u^7) \, du$$

$$= \frac{u^6}{6} - \frac{u^8}{8} + C = \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C.$$

4. (6 points) Evaluate the following definite integral exactly. In case of an improper integral, determine the convergence of the integral. Show clearly any limit computation you do. If the integral converges, find its value. (You may use formulas 1-18, 39-42, 50, 51 only from the table of integrals for this problem.)

$$\int_0^1 \frac{\ln x}{20x} \, dx$$

Improper at $x = 0$.

$$\int_0^1 \frac{\ln x}{20x} dx = \lim_{t \to 0^+} \int_t^1 \frac{\ln x}{20x} dx.$$

$$\int \frac{\ln x}{20x} dx \underset{\substack{u = \ln x \\ du = \frac{1}{x} dx}}{=\!=\!=} \frac{1}{20} \int u \, du = \frac{1}{20} \frac{u^2}{2} + C = \frac{(\ln x)^2}{40} + C.$$

$$\lim_{t \to 0^+} \int_t^1 \frac{\ln x}{20x} dx = \lim_{t \to 0^+} \left[\frac{(\ln x)^2}{40} \right]_t^1 = \lim_{t \to 0^+} \left[0 - \frac{(\ln t)^2}{40} \right].$$

As $t \to 0^+$, $\ln t \to -\infty$. So $(\ln t)^2 \to \infty$.

So $\lim_{t \to 0^+} \int_t^1 \frac{\ln x}{20x} dx = \infty$.

Hence $\int_0^1 \frac{\ln x}{20x} dx$ diverges.

5. (6 points) Evaluate the following definite integral exactly. In case of an improper integral, determine the convergence of the integral. Show clearly any limit computation you do. If the integral converges, find its value. (You may use formulas 1-18, 39-42, 50, 51 only from the table of integrals for this problem.)

$$\int_0^{\pi/2} 5t\cos(2t)\,dt$$

Use integration by parts to find $\int 5t\cos(2t)\,dt$.

$u = 5t \qquad dv = \cos(2t)$

$du = 5\,dt \qquad v = \int \cos(2t)\,dt = \dfrac{\sin(2t)}{2}$.

$$\int 5t\cos(2t)\,dt = 5t\frac{\sin(2t)}{2} - \int \frac{\sin(2t)}{2}\,5\,dt$$

$$= \frac{5}{2}t\sin(2t) - \frac{5}{2}\int \sin(2t)\,dt$$

$$= \frac{5}{2}t\sin(2t) + \frac{5}{2}\,\frac{\cos(2t)}{2} + C.$$

$$\int_0^{\pi/2} 5t\cos(2t)\,dt = \frac{5}{2}t\sin(2t) + \frac{5}{4}\cos(2t)\Big]_0^{\pi/2}.$$

$$= 0 + \frac{5}{4}(-1) - 0 - \frac{5}{4}$$

$$= -\frac{10}{4} = \frac{-5}{2}.$$

6. (5 points) Use comparisons to determine the convergence of the following integral.

$$\int_3^\infty \frac{2x^2+7}{\sqrt{9x^5-x}}\,dx\ .$$

$$9x^5 - x < 9x^5 \quad , \text{ for } x \geq 3 .$$

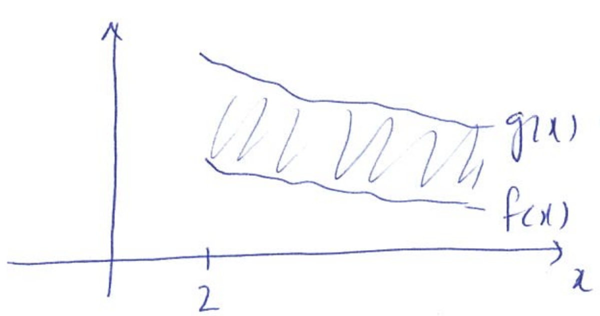
$$\frac{1}{\sqrt{9x^5-x}} > \frac{1}{\sqrt{9x^5}} .$$

$$\frac{2x^2+7}{\sqrt{9x^5-x}} > \frac{2x^2}{\sqrt{9x^5}} = \frac{2}{3}\frac{1}{\sqrt{x}} .$$

$\int_3^\infty \frac{2}{3}\frac{1}{\sqrt{x}}\,dx$ diverges $(p = \frac{1}{2} < 1)$.

So $\int_3^\infty \frac{2x^2+7}{\sqrt{9x^5-x}}$ diverges .

7. (3 points) Suppose $f(x)$ and $g(x)$ are two functions such that $0 \leq f(x) \leq g(x)$ over the interval $[2, \infty)$. Further suppose that $\int_2^\infty f(x)\,dx$ converges and $\int_2^\infty g(x)\,dx$ diverges. Can you conclude whether the area between $f(x)$ and $g(x)$ over the interval $[2, \infty)$ is either finite or infinite? Or is it not possible to say anything about the area with certainty? Explain your answer.



Since $\int_2^\infty f(x)\,dx$ converges,
area under f(x) over $[2,\infty)$ is finite.
Since $\int_2^\infty g(x)\,dx$ diverges,
area under g(x) over $[2,\infty)$ is infinite.

Area between f(x) and g(x) over $[2,\infty)$
$= \underbrace{\text{area under } g(x)}_{\text{infinite}} - \underbrace{\text{area under } f(x)}_{\text{finite}} = \text{infinite}$.

8. (6 points) The amount of a certain product (in tons) used each day in a manufacturing plant is a continuous random variable, X, with the probability density function given by $f(x) = 0.25e^{-0.25x}$ for $x \geq 0$ (the function is zero for all other values of x).

(a) What is the probability that the manufacturing plant will use more than 4 tons of the product on a randomly selected day?

$$\text{Probability} = \int_4^{\infty} 0.25e^{-0.25x}\,dx = \lim_{t\to\infty}\int_4^t 0.25e^{-0.25x}\,dx$$

$$\int 0.25e^{-0.25x}\,dx = -e^{-0.25x} + C \quad (u = -0.25x,\ du = -0.25\,dx).$$

$$\lim_{t\to\infty}\int_4^t 0.25e^{-0.25x}\,dx = \lim_{t\to\infty}\left[-e^{-0.25x}\right]_4^t$$

$$= \lim_{t\to\infty}\left[-e^{-0.25t} + e^{-1}\right]$$

As $t\to\infty$, $e^{-0.25t} \to 0$.

So $\lim_{t\to\infty}\int_4^{\infty} 0.25e^{-0.25x}\,dx = 0 + \frac{1}{e} = \frac{1}{e}$.

(b) What does the integral $\int_0^2 f(x)\,dx$ compute? Write this in words (without using X in your sentence(s)). (Do not evaluate the integral).

The integral computes the probability that the manufacturing plant will use at most 2 tons of the product on a randomly selected day.

9. (6 points) Suppose you have the following information for a function $g(x)$.

$$g(1) = 10, \ g'(1) = -3, \ g''(1) = 0, \ g'''(1) = 5 \text{ and } |g^{(4)}(x)| \le x^2 \text{ for all } x \text{ in } [-2, 1.5].$$

(a) Use a third order Taylor polynomial to estimate $g(-0.5)$.

$$P_3(x) = g(1) + g'(1)(x-1) + \frac{g''(1)}{2!}(x-1)^2 + \frac{g'''(1)}{3!}(x-1)^3$$

$$= 10 - 3(x-1) + 0 + \frac{5}{6}(x-1)^3 .$$

This is the only polynomial we can write with the given information.

$$g(-0.5) \approx P_3(-0.5)$$

$$= 10 - 3(-0.5-1) + \frac{5}{6}(-0.5-1)^3$$

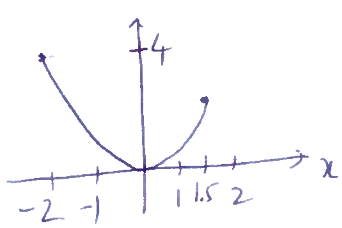
$$= +11.6875$$

(b) Use Taylor's theorem to find an upper bound on the error in your estimate in part (a). To find the best possible K_{n+1}, use the interval $[-2, 1.5]$.

By Taylor's theorem,

$$|P_3(-0.5) - g(-0.5)| \le \frac{K_4}{4!}|-0.5-1|^4$$

Since $|g^{(4)}(x)| \le x^2$ for all x in $[-2, 1.5]$, we take the maximum value of x^2 on $[-2, 1.5]$ as K_4.



Graph of x^2.

$K_4 = 4$

So $$|P_3(-0.5) - g(-0.5)| \le \frac{4}{4!}|-1.5|^4$$

i.e. $$\underbrace{|P_3(-0.5) - g(-0.5)|}_{\text{error in estimate}} \le \underbrace{0.84375}_{\text{upper bound on error}}$$