# MATH106A,B CALCULUS II - PROF. P. WONG

EXAM II - MARCH 7, 2014

**NAME:**

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

*Advice*: DON'T spend too much time on a single problem.

| **Problems** | **Maximum Score** | **Your Score** |
|---|---|---|
| 1. | 20 | |
| 2. | 20 | |
| 3. | 20 | |
| 4. | 20 | |
| 5. | 20 | |
| **Total** | 100 | |

**1.** Evaluate each of the following indefinite integrals (be sure to indicate what techniques you use).

(10 pts.)(a)

$$\int x^2 \cos(2x^3)\ dx.$$

**Let** $u = 2x^3$**. Then** $du = 6x^2\ dx$ **or** $x^2\ dx = \frac{1}{6}\ du$**. It follows that**

$$\begin{aligned}\int x^2 \cos(2x^3)\ dx &= \int \cos(2x^3) \cdot x^2\ dx \\ &= \int \cos u \cdot \frac{1}{6}\ du \\ &= \frac{1}{6}\sin u + C \\ &= \frac{\sin(2x^3)}{6} + C.\end{aligned}$$

(10 pts.)(b)

$$\int \frac{\ln x\ dx}{\sqrt{x}}.$$

**Let** $u = \ln x$ **and** $dv = x^{-1/2}\ dx$**. Then,** $v = 2x^{1/2}$ **and** $du = \frac{dx}{x}$**. It follows that**

$$\begin{aligned}\int \frac{\ln x\ dx}{\sqrt{x}} &\overset{\textbf{IBP}}{=} \ln x \cdot 2x^{1/2} - \int 2x^{1/2} \cdot \frac{1}{x}\ dx \\ &= 2\sqrt{x}\ln x - 2\int x^{-1/2}\ dx \\ &= 2\sqrt{x}\ln x - 2\frac{x^{1/2}}{1/2} + C \\ &= 2\sqrt{x}\ln x - 4\sqrt{x} + C = 2\sqrt{x}(\ln x - 2) + C.\end{aligned}$$

**2.** Evaluate each of the following indefinite integrals (be sure to indicate what techniques you use).

(10 pts.)(a)

$$\int \frac{dt}{t^2\sqrt{t^2-1}}.$$

**Let** $t = \sec\theta$ **so that** $t^2 = \sec^2\theta, \sqrt{t^2-1} = \tan\theta$ **and** $dt = \sec\theta\tan\theta\ d\theta$**. It follows that**

$$\begin{aligned}\int \frac{dt}{t^2\sqrt{t^2-1}} &= \int \frac{\sec\theta\tan\theta\ d\theta}{\sec^2\theta\tan\theta} \\ &= \int \cos\theta\ d\theta = \sin\theta + C \\ &= \frac{\sqrt{t^2-1}}{t} + C.\end{aligned}$$

**That** $\sin\theta = \frac{\sqrt{t^2-1}}{t}$ **follows from the right angled triangle associated to the substitution** $t = \sec\theta$**.**

(10 pts.)(b)

$$\int \frac{x^3+2x^2-3x+4}{x^2+2x-3}\ dx.$$

**First, using long division, we have** $\frac{x^3+2x^2-3x+4}{x^2+2x-3} = x + \frac{4}{x^2+2x-3}$**. Now, we write**

$$\frac{4}{x^2+2x-3} = \frac{4}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}.$$

**Thus,**

$$4 \equiv A(x-1) + B(x+3).$$

**At** $x = 1$**, we have** $4 = 4B \Rightarrow B = 1$**. At** $x = -3$**, we have** $4 = -4A \Rightarrow A = -1$**. It foolows that**

$$\begin{aligned}\int \frac{x^3+2x^2-3x+4}{x^2+2x-3}\ dx &= \int x - \frac{1}{x+3} + \frac{1}{x-1}\ dx \\ &= \frac{x^2}{2} - \ln|x+3| + \ln|x-1| + C.\end{aligned}$$

**3.** Evaluate each of the following improper integrals.

(10 pts.)(a)

$$\int_{-\infty}^{1} \frac{dx}{(2x-3)^3}$$

**First,**

$$\int_{-\infty}^{1} \frac{dx}{(2x-3)^3} = \lim_{b\to-\infty} \int_{b}^{1} \frac{dx}{(2x-3)^3}.$$

**Let** $u = 2x - 3$ **so that** $du = 2\ dx$ **or** $dx = \frac{du}{2}$**. Now**

$$\begin{aligned}\int \frac{dx}{(2x-3)^3} &= \frac{1}{2}\int u^{-3}\ du \\ &= \frac{1}{2}\cdot\frac{u^{-2}}{-2} + C = -\frac{1}{4}u^{-2} + C \\ &= -\frac{1}{4}(2x-3)^{-2} + C.\end{aligned}$$

**This implies that**

$$\int_{-\infty}^{1} \frac{dx}{(2x-3)^3} = \lim_{b\to-\infty} -\frac{1}{4}(2x-3)^{-2}\Big|_{b}^{1} = \lim_{b\to-\infty} -\frac{1}{4} + \frac{1}{4}(2b-3)^{-2} = -\frac{1}{4}.$$

(10 pts.)(b)

$$\int_{2}^{3} \frac{x}{\sqrt{3-x}}\ dx$$

**Let** $u = 3 - x$**. Thus,** $du = -dx$ **and** $x = 3 - u$**. It follows that**

$$\begin{aligned}\int_{2}^{3} \frac{x}{\sqrt{3-x}}\ dx = \int_{-1}^{0} -\frac{3-u}{\sqrt{u}}\ du &= \int_{0}^{1} \frac{3-u}{\sqrt{u}}\ du \\ &= \lim_{b\to0} \int_{b}^{1} 3u^{-1/2} - u^{1/2} \\ &= \lim_{b\to0} \frac{3u^{1/2}}{1/2} - \frac{u^{3/2}}{3/2}\Big|_{b}^{1} \\ &= \lim_{b\to0} \left(6 - \frac{2}{3}\right) - \left(\frac{3b^{1/2}}{1/2} - \frac{b^{3/2}}{3/2}\right) \\ &= 6 - \frac{2}{3} = \frac{16}{3}.\end{aligned}$$

**4.** Let $f(x) = e^{2x}$.

(8 pts.)(a) Find the third-order Taylor polynomial $P_3(x)$ of $f(x)$ based at $x_0 = 1$.

**Note that** $f(x) = e^{2x}, f'(x) = 2e^{2x}, f''(x) = 4e^{2x}$ **and** $f'''(x) = 8e^{2x}$**. It follows that** $f(1) = e^2, f'(1) = 2e^2, f''(1) = 4e^2$ **and** $f'''(1) = 8e^2$**. Thus the desired Taylor polynomial is given by**

$$\begin{aligned} P_3(x) &= e^2 + 2e^2(x-1) + \frac{4e^2}{2}(x-1)^2 + \frac{8e^2}{3!}(x-1)^3 \\ &= e^2[1 + 2(x-1) + 2(x-1)^2 + \frac{4}{3}(x-1)^3]. \end{aligned}$$

(8 pts.)(b) Find the third-order Maclaurin polynomial $M_3(x)$ of $f(x)$.

**Using the calculation of the first three derivatives of** $f$ **from part (a), we have** $f(0) = 1, f'(0) = 2, f''(0) = 4, f'''(0) = 8$**. It follows that**

$$M_3(x) = 1 + 2x + \frac{4}{2}x^2 + \frac{8}{3!}x^3 = 1 + 2x + 2x^2 + \frac{4}{3}x^3.$$

(4 pts.)(c) What is the maximum error committed by using $M_3(x)$ (as in part (b)) over the interval $[-1, 1]$, according to Taylor's Theorem? [Hint: how do you obtain $K_4$?]

**By Taylor's theorem, we know that**

$$|M_3(x) - f(x)| \leq \frac{K_4}{4!}|x - 0|^4$$

**where** $K_4$ **is a constant such that** $|f^{(4)}(x)| \leq K_4$ **for all** $x$**,** $-1 \leq x \leq 1$**. Note that** $f'''(x) = 8e^{2x}$ **thus** $f^{(4)}(x) = 16e^{2x}$**. Therefore, over the interval** $[-1, 1]$**,** $|f^{(4)}(x)| \leq 16e^2$ **so that we can choose** $K_4 = 16e^2$**. We can conclude that the maximum error committed by** $M_3$**, using Taylor's theorem, is**

$$\frac{K_4}{4!} \cdot 1 = \frac{16e^2}{24} = \frac{2e^2}{3}.$$

**5.** (12 pts.)(a) Use comparison to determine whether the following improper integral converges or diverges. Justify your answer.

$$\int_1^\infty \frac{e^{-x}}{\sqrt{x^3+1}}\ dx$$

**For** $x \geq 1, \sqrt{x^3+1} > 1$ **so that** $\frac{1}{\sqrt{x^3+1}} < 1$**. It follows that**

$$\frac{e^{-x}}{\sqrt{x^3+1}} < e^{-x} \qquad \textbf{and} \qquad \int_1^\infty \frac{e^{-x}}{\sqrt{x^3+1}}\ dx < \int_1^\infty e^{-x}\ dx, \textbf{which converges}.$$

**This comparison implies that the improper integral** $\displaystyle\int_1^\infty \frac{e^{-x}}{\sqrt{x^3+1}}\ dx$ **converges.**

(8 pts.)(b) Consider the following function

$$f(x) = \begin{cases} kx^2(12-x), & \text{for } 0 \leq x \leq 12; \\ 0, & \text{otherwise.} \end{cases}$$

For what value of $k$ is $f(x)$ a *probability density function*?

**For** $f(x)$ **to a p.d.f., we must have (1)** $f(x) \geq 0$ **and (2)** $\displaystyle\int_{-\infty}^\infty f(x)\ dx = 1$**. Note that**

$$\begin{aligned} \int_{-\infty}^\infty f(x)\ dx &= \int_0^{12} kx^2(12-x)\ dx \\ &= k\int_0^{12} 12x^2 - x^3\ dx = k \cdot \left(4x^3 - \frac{x^4}{4}\right)\Bigg|_0^{12} \\ &= k \cdot \left(4 \cdot (12)^3 - \frac{(12)^4}{4}\right) = (12)^3 k. \end{aligned}$$

**Thus, for** $f(x)$ **to be a p.d.f.,** $(12)^3 k = 1$ **or** $k = \frac{1}{(12)^3}$**.**