

Math 106 Winter 2014

Test 2 (50 points)

Name: Solutions

Show all your work to receive full credit for a problem. Points will be taken off if you do not show how you arrived at your answer, even if the final answer is correct.

Please keep your written answers brief; be clear and to the point. Points will be taken off for rambling and for incorrect or irrelevant statements.

Do not use the calculator integral function. Whenever possible, find the exact values of integrals by finding antiderivatives or using the table of integrals.

When you use a formula from the table of integrals, mention the formula number and the value(s) of any constant(s) that you may need.

Give exact answers. If needed, round off your answers to four decimal places.

Include units in your answers wherever possible.

There are seven questions. Questions are printed on both sides of a page.

You may use any of the following facts:

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!} |x - x_0|^{n+1}$$

$$\int u \, dv = uv - \int v \, du$$

$$\int_1^{\infty} \frac{1}{x^p} \, dx \text{ converges for } p > 1 \text{ and diverges for } p \leq 1.$$

$$\int_0^{\infty} e^{-ax} \, dx \text{ converges for } a > 0.$$

1. (8 points) Evaluate the following integral. (You may use formulas 1-18, 39-42, 50, 51 only from the table of integrals for this problem.)

$$\int \frac{x^2}{\sqrt{9-x^2}} dx$$

Let  $x = 3 \sin t$

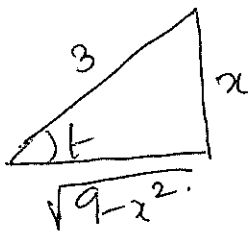
$$\frac{dx}{dt} = 3 \cos t \cdot dx = 3 \cos t dt$$

$$\int \frac{x^2}{\sqrt{9-x^2}} dx = \int \frac{9 \sin^2 t}{\sqrt{9-9 \sin^2 t}} \cdot 3 \cos t dt$$

$$= \int \frac{27 \sin^2 t \cos t}{3 \cdot \cos t} dt$$

$$= 9 \int \sin^2 t dt \quad \text{Formula 40, } n=2, a=1.$$

$$= 9 \left[ \frac{-\sin t \cos t}{2} + \frac{t}{2} \right].$$



$$\sin t = \frac{x}{3} \quad t = \arcsin\left(\frac{x}{3}\right)$$

$$\text{So } \cos t = \frac{\sqrt{9-x^2}}{3}$$

$$\int \frac{x^2}{\sqrt{9-x^2}} dx = \frac{9}{2} \left[ t - \sin t \cos t \right]$$

$$= \frac{9}{2} \left[ \arcsin\left(\frac{x}{3}\right) - \frac{x}{3} \cdot \frac{\sqrt{9-x^2}}{3} \right]$$

$$= \frac{9}{2} \arcsin\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C.$$

2. (10 points) Evaluate each of the following integrals. (You may use formulas 1-18, 39-42, 50, 51 only from the table of integrals for this problem.)

(a)  $\int \sec^3 x \tan^3 x \, dx.$

Let  $u = \sec x$ ,  $du = \sec x \tan x \, dx.$

$$\int \sec^3 x \tan^3 x \, dx$$

$$= \int \sec^2 x \tan^2 x \cdot \sec x \tan x \, dx$$

$$= \int \sec^2 x \cdot (\sec^2 x - 1) \sec x \tan x \, dx$$

$$= \int u^2 (u^2 - 1) \, du$$

$$= \int (u^4 - u^2) \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} = \frac{(\sec x)^5}{5} - \frac{(\sec x)^3}{3} + C.$$

(b)  $\int (7 - 5x) \ln x \, dx.$

Integration by parts.

Let  $u = \ln x$ ,  $dv = 7 - 5x.$

$$du = \frac{1}{x} \, dx, \quad v = \int (7 - 5x) \, dx = 7x - \frac{5x^2}{2}.$$

$$\int (7 - 5x) \ln x \, dx = (\ln x) \left( 7x - \frac{5x^2}{2} \right) - \int \left( 7x - \frac{5x^2}{2} \right) \cdot \frac{1}{x} \, dx$$

$$= (\ln x) \left( 7x - \frac{5x^2}{2} \right) - \int \left( 7 - \frac{5}{2}x \right) \, dx$$

$$= (\ln x) \left( 7x - \frac{5x^2}{2} \right) - \left( 7x - \frac{5x^2}{4} \right) + C.$$

3. (7 points) Evaluate the following integral. (You may use formulas 1-18 only from the table of integrals for this problem.)

$$\int \frac{x^2 + 2x - 1}{(3-x)(x^2+5)} dx$$

Partial fractions.

$$\frac{x^2 + 2x - 1}{(3-x)(x^2+5)} = \frac{Ax+B}{x^2+5} + \frac{C}{3-x}$$

$$x^2 + 2x - 1 = (Ax+B)(3-x) + C(x^2+5)$$

$$\underline{x=3}: \quad 9+6-1 = 0+14C \quad \boxed{C=1}$$

$$\underline{x=0}: \quad -1 = \cancel{B \cdot 2} B \cdot 3 + 5 \quad \boxed{B=-2}$$

$$\underline{x=1}: \quad 1+2-1 = (A-2) \cdot 2 + 6 \quad \boxed{A=0}$$

$$\int \frac{x^2 + 2x - 1}{(3-x)(x^2+5)} dx = \int \frac{-2}{x^2+5} dx + \int \frac{1}{3-x} dx$$

$$= \frac{-2}{\sqrt{5}} \arctan\left(\frac{x}{\sqrt{5}}\right) + \int \frac{1}{3-x} dx$$

$$\text{formula 13, } a=\sqrt{5} \quad \begin{array}{l} u=3-x \\ du=-dx \end{array}$$

$$= \frac{-2}{\sqrt{5}} \arctan\left(\frac{x}{\sqrt{5}}\right) - \int \frac{1}{u} du$$

$$= \frac{-2}{\sqrt{5}} \arctan\left(\frac{x}{\sqrt{5}}\right) - \ln|3-x| + C$$

4. (7 points) Evaluate the following definite integral exactly. In case of an improper integral, determine the convergence of the integral. Show clearly any limit computation you do. If the integral converges, find its value. (You may use formulas 1-18, 39-42, 50, 51 only from the table of integrals for this problem.)

$$\int_{1.5}^{5.5} \frac{dx}{11-2x}$$

Integral is improper at 5.5.

$$\int \frac{1}{11-2x} dx \quad \begin{array}{l} \underline{\underline{u=11-2x}} \\ du = -2 dx \end{array} \quad \int \frac{1}{u} \cdot \frac{du}{-2} = -\frac{1}{2} \ln|u| = -\frac{1}{2} \ln|11-2x| + C$$

$$\int_{1.5}^b \frac{dx}{11-2x} = \left. -\frac{1}{2} \ln|11-2x| \right]_{1.5}^b$$

$$= -\frac{1}{2} \ln|11-2b| + \frac{1}{2} \ln 8$$

$$\lim_{b \rightarrow 5.5^-} \int_{1.5}^b \frac{dx}{11-2x} = \lim_{b \rightarrow 5.5^-} \left[ -\frac{1}{2} \ln|11-2b| + \frac{1}{2} \ln 8 \right]$$

As  $b \rightarrow 5.5^-$ ,  $11-2b \rightarrow 0^+$ . or look at graph of  
 $\ln|11-2b| \rightarrow -\infty$ .  $-\frac{1}{2} \ln|11-2b|$   
 $-\frac{1}{2} \ln|11-2b| \rightarrow +\infty$ .

Hence,  $\lim_{b \rightarrow 5.5^-} \left[ -\frac{1}{2} \ln|11-2b| + \frac{1}{2} \ln 8 \right] = \infty$ .

So  $\int_{1.5}^{5.5} \frac{dx}{11-2x}$  diverges.

5. (5 points) Use comparison to determine the convergence of the following integral.

$$\int_5^{\infty} \frac{9x^2 - x - 1}{6x^4 + 17} dx$$

$$6x^4 + 17 \geq 6x^4 \text{ for } x \geq 5.$$

$$\frac{1}{6x^4 + 17} \leq \frac{1}{6x^4}$$

$$\frac{9x^2 - x - 1}{6x^4 + 17} \leq \frac{9x^2}{6x^4} \text{ (since } 9x^2 - x - 1 \leq 9x^2 \text{)}.$$

$$\int_5^{\infty} \frac{9x^2}{6x^4} dx = \int_5^{\infty} \frac{3}{2} \frac{1}{x^2} dx = \frac{3}{2} \int_5^{\infty} \frac{1}{x^2} dx \text{ converges.}$$

( $p = 2 > 1$ ).

So by comparison test,

$$\int_5^{\infty} \frac{9x^2 - x - 1}{6x^4 + 17} dx \text{ converges.}$$

6. (6 points) Let  $X$  denote the width in millimeters (mm) of metal pipes from an automated production line. The p.d.f. of  $X$  is given by  $f(x) = 7e^{-7(x-5)}$  for  $x \geq 5$  (the function is zero for all other values of  $x$ ). Find the probability that a randomly selected metal pipe has a width of at least 6 mm.

$$\text{Required probability} = \int_6^{\infty} f(x) dx$$

$$= 1 - \int_5^6 f(x) dx \quad \left( \text{since } \int_5^{\infty} f(x) dx = 1 \right. \\ \left. \text{because } f(x) \text{ is a p.d.f.} \right)$$

$$\begin{aligned} \int f(x) dx &= \int 7e^{-7(x-5)} dx & u &= -7(x-5) \cdot d \\ & & du &= -7 dx \cdot \\ &= \int 7e^u \cdot \frac{du}{-7} \\ &= -\int e^u du = -e^u = -e^{-7(x-5)} + C. \end{aligned}$$

$$\int_5^6 f(x) dx = -e^{-7(x-5)} \Big|_5^6 = -e^{-7} + 1.$$

$$1 - \int_5^6 f(x) dx = 1 - (-e^{-7} + 1) = e^{-7}.$$

7. (7 points) Suppose the third-order Taylor polynomial for a function  $f$  based at  $x_0 = -1$  is

$$P_3(x) = 7 - 3(x+1) + \frac{2}{5}(x+1)^3.$$

(a) Find  $f''(-1)$ .

Since coefficient of  $(x+1)^2$  in  $P_3(x)$  is 0,

$$f''(-1) = 0.$$

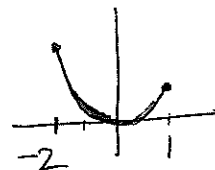
(b) Use Taylor's theorem to find the best possible upper bound on the error in estimating  $f(0.1)$  using  $P_3$ . Assume that  $|f^{(4)}(x)| \leq x^2$  for all  $x$  in  $[-2, 1]$ .

By Taylor's theorem,

$$|f(0.1) - P_3(0.1)| \leq \frac{K_4}{4!} |0.1 - (-1)|^4.$$

$$|f^{(4)}(x)| \leq x^2 \text{ for all } x \text{ in } [-2, 1].$$

$$\text{For } x \text{ in } [-2, 1], \max x^2 = 4.$$



$$\text{So } K_4 = 4.$$

$$|f(0.1) - P_3(0.1)| \leq \frac{4}{4!} (1.1)^4 = \frac{(1.1)^4}{6} = 0.2440$$