Math 106: Review for Exam II - SOLUTIONS

INTEGRATION TIPS

- Substitution: usually let $u =$ a function that's "inside" another function, especially if du (possibly off by a multiplying constant) is also present in the integrand.
- Parts: $\int u\,dv = uv - \int v\,du$ or $\int uv'\,dx = uv - \int u'v\,dx$

  How to choose which part is u? Let u be the part that is higher up in the **LIATE** mnemonic below. (The mnemonics **ILATE** and **LIPET** will work equally well if you have learned one of those instead; in the latter **A** is replaced by **P**, which stands for "polynomial.")

  Logarithms (such as $\ln x$)

  Inverse trig (such as $\arctan x, \arcsin x$)

  Algebraic (such as x, x^2, x^3+4)

  Trig (such as $\sin x, \cos 2x$)

  Exponentials (such as e^x, e^{3x})

- Rational Functions (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

  Partial Fractions: here's an illustrative example of the setup.

$$\frac{3x^2+11}{(x+1)(x-3)^2(x^2+5)} = \frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{Dx+E}{x^2+5}$$

  Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor $(x-3)$ on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term ($Dx+E$ here) above it on the right.

- Trigonometric Substitutions: some suggested substitutions and useful formulae follow.

Radical Form	$\sqrt{a^2-x^2}$	$\sqrt{a^2+x^2}$	$\sqrt{x^2-a^2}$
Substitution	$x = a\sin t$	$x = a\tan t$	$x = a\sec t$

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x$$

$$\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2} \qquad \cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2}$$

$$\sin(2x) = 2\sin x\cos x$$

- Powers of Trigonometric Functions: here are some strategies for dealing with these.

$\int \sin^m x \cos^n x\ dx$	Possible Strategy	Identity to Use
m odd	Break off one factor of $\sin x$ and substitute $u = \cos x$.	$\sin^2 x = 1 - \cos^2 x$
n odd	Break off one factor of $\cos x$ and substitute $u = \sin x$.	$\cos^2 x = 1 - \sin^2 x$
m even AND n even	Use $\sin^2 x + \cos^2 x = 1$ to reduce to only powers of $\sin x$ or only powers of $\cos x$, then use table of integrals #39–42 or identities shown to right of this box.	$\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2}$ $\cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2}$

$\int \tan^m x \sec^n x \, dx$	Possible Strategy	Identity to Use
m odd	Break off one factor of $\sec x \tan x$ and substitute $u = \sec x$.	$\tan^2 x = \sec^2 x - 1$
n even	Break off one factor of $\sec^2 x$ and substitute $u = \tan x$.	$\sec^2 x = \tan^2 x + 1$
m even AND n odd	Use identity at right to reduce to powers of $\sec x$ alone. Then use table of integrals #51 or integration by parts.	$\tan^2 x = \sec^2 x - 1$

Useful Trigonometric Derivatives and Antiderivatives

$\frac{d}{dx} \tan x = \sec^2 x$ $\qquad$ $\frac{d}{dx} \sec x = \sec x \tan x$ $\qquad$ $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

- Improper integrals: look for ∞ as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration. It may be useful to know the following limits.

  $\lim_{x\to\infty} e^x = \infty$

  $\lim_{x\to\infty} e^{-x} = 0$ $\qquad$ Note: this is the same as $\lim_{x\to-\infty} e^x$.

  $\lim_{x\to\infty} 1/x = 0$ $\qquad$ Note: the answer is the same for $\lim_{x\to\infty} 1/x^2$ and similar functions.

  $\lim_{x\to 0^+} 1/x = \infty$ $\qquad$ Note: the answer is the same for $\lim_{x\to 0^+} 1/x^2$ and similar functions.

  $\lim_{x\to\infty} \ln x = \infty$

  $\lim_{x\to 0^+} \ln x = -\infty$

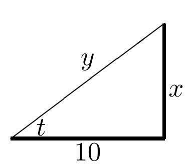
  $\lim_{x\to\infty} \arctan x = \pi/2$

1. **Evaluate the following.**

   (a) Let $u = \sin x$, so $du = \cos x \, dx$.

$$\begin{aligned}\int \sin^6 x \cos^3 x \, dx &= \int \sin^6 x(1-\sin^2 x)\cos x \, dx \qquad \text{Use } \cos^2 x = 1 - \sin^2 x.\\ &= \int u^6(1-u^2)\, du\\ &= \int (u^6 - u^8)\, du\\ &= \frac{u^7}{7} - \frac{u^9}{9} + C\\ &= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C\end{aligned}$$

   (b) Let $x = 10\tan t$, so $dx = 10\sec^2 t \, dt$.



$$x^2 + 10^2 = y^2 \Rightarrow y = \sqrt{x^2+100}$$

$$\sec t = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{x^2+100}}{10}$$

$$\tan t = \frac{\text{opp}}{\text{adj}} = \frac{x}{10}$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{100+x^2}} &= \int \frac{10\sec^2 t\ dt}{\sqrt{100+100\tan^2 t}} \\
&= \int \frac{10\sec^2 t\ dt}{10\sqrt{1+\tan^2 t}} && \text{Now use } 1+\tan^2 t = \sec^2 t. \\
&= \int \frac{\sec^2 t\ dt}{\sqrt{\sec^2 t}} \\
&= \int \sec t\ dt \\
&= \ln|\sec t + \tan t| + C && \text{Now use triangle above.} \\
&= \ln\left|\frac{\sqrt{x^2+100}}{10} + \frac{x}{10}\right| + C
\end{aligned}$$

(c) This is an improper integral, so we need to use a limit.

$$\begin{aligned}
\int_3^\infty \frac{1}{x(\ln x)^{100}}\ dx &= \lim_{t\to\infty} \int_3^t \frac{1}{x(\ln x)^{100}}\ dx \\
&= \lim_{t\to\infty} \int_{x=3}^{x=t} \frac{1}{u^{100}}\ du && \text{Substitute } u = \ln x, \text{so } du = \frac{dx}{x}. \\
&= \lim_{t\to\infty} \frac{u^{-99}}{-99}\Big|_{x=3}^{x=t} \\
&= \lim_{t\to\infty} \frac{-1}{99(\ln x)^{99}}\Big|_3^t \\
&= \lim_{t\to\infty} \left[\frac{-1}{99(\ln t)^{99}} - \frac{-1}{99(\ln 3)^{99}}\right] \\
&= 0 - \frac{-1}{99(\ln 3)^{99}} \\
&= \frac{1}{99(\ln 3)^{99}} && \text{So, the integral converges (to this value).}
\end{aligned}$$

(d) We'll use integration by parts: $u = x \Rightarrow du = dx$ and $dv = e^{-2x} \Rightarrow v = \dfrac{e^{-2x}}{-2}$.

$$\begin{aligned}
\int_0^\infty xe^{-2x}\ dx &= \lim_{t\to\infty} \int_0^t xe^{-2x}\ dx \\
&= \lim_{t\to\infty} \left[x\frac{e^{-2x}}{-2}\Big|_0^t - \int_0^t \frac{e^{-2x}}{-2}\ dx\right] \\
&= \lim_{t\to\infty} \left[x\frac{e^{-2x}}{-2} - \frac{e^{-2x}}{4}\right]_0^t \\
&= \lim_{t\to\infty} \left[\frac{-x}{2e^{2x}} - \frac{1}{4e^{2x}}\right]_0^t \\
&= \lim_{t\to\infty} \left[\frac{-t}{2e^{2t}} - \frac{1}{4e^{2t}}\right] - \left[\frac{0}{2e^0} - \frac{1}{4e^0}\right] \\
&=(0-0) - (0-1/4) \\
&=1/4 && \text{So, the integral converges (to this value).}
\end{aligned}$$

(e) Partial Fractions:

Write $\dfrac{3x^2+2x-13}{(x-3)(x^2+1)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-3}$. Now multiply both sides by $(x-3)(x^2+1)$ to get

$$3x^2+2x-13=(Ax+B)(x-3)+C(x^2+1).$$

Let $x=3$. Then $20=C(10)$, so $C=2$.
Let $x=0$. Then $-13=B(-3)+2(1)$, so $B=5$.
Let $x=1$. Then $-8=(A(1)+5)(-2)+2(2)$, so $A=1$.

$$\begin{aligned}\int \frac{3x^2+2x-13}{(x-3)(x^2+1)}\,dx &= \int \left[\frac{x+5}{x^2+1}+\frac{2}{x-3}\right]\,dx\\ &= \int \left[\frac{x}{x^2+1}+\frac{5}{x^2+1}+\frac{2}{x-3}\right]\,dx \qquad \text{Let } u=x^2+1, \text{ so } du=2xdx.\\ &= \int \frac{\frac{1}{2}du}{u}+\int \left[\frac{5}{x^2+1}+\frac{2}{x-3}\right]\,dx\\ &= \frac{\ln u}{2}+5\arctan x+2\ln|x-3|+D\\ &= \frac{\ln(x^2+1)}{2}+5\arctan x+2\ln|x-3|+D\end{aligned}$$

(f) Since the degree of the numerator is greater than or equal to the degree of the denominator, we do long division.

$$\begin{array}{r|l} & 4x^2 \quad -3x+ \quad 2+\dfrac{-5}{x-6}\\ \hline x-6 & 4x^3-27x^2+20x-17\\ & \underline{4x^3-24x^2}\\ & \quad -3x^2\\ & \quad \underline{-3x^2+18x}\\ & \qquad\qquad 2x\\ & \qquad\qquad \underline{2x-12}\\ & \qquad\qquad\quad -5\end{array}$$

Now, we compute the integral.

$$\int \frac{4x^3-27x^2+20x-17}{x-6}\,dx = \int \left[4x^2-3x+2-\frac{5}{x-6}\right]dx = \frac{4x^3}{3}-\frac{3x^2}{2}+2x-5\ln|x-6|+C$$

(g) This integral is improper at $x=1$ because the integrand has a vertical asymptote there, so we split into two integrals.

$$\begin{aligned}\boldsymbol{\int_{-1}^{5}\frac{1}{(x-1)^6}\,dx} &= \int_{-1}^{1}\frac{dx}{(x-1)^6}+\int_{1}^{5}\frac{dx}{(x-1)^6}\\ &= \lim_{a\to1^-}\int_{-1}^{a}\frac{dx}{(x-1)^6}+\lim_{b\to1^+}\int_{b}^{5}\frac{dx}{(x-1)^6}\\ &= \lim_{a\to1^-}\left.\frac{-1}{5(x-1)^5}\right|_{-1}^{a}+\lim_{b\to1^+}\left.\frac{-1}{5(x-1)^5}\right|_{b}^{5} \qquad \int u^{-6}\,du=\frac{u^{-5}}{-5}+C\\ &= \lim_{a\to1^-}\left[\frac{-1}{5(a-1)^5}-\frac{-1}{5(-1-1)^5}\right]+\lim_{b\to1^+}\left[\frac{-1}{5(5-1)^5}-\frac{-1}{5(b-1)^5}\right]\end{aligned}$$

Since $\lim_{a\to1^-}\frac{-1}{5(a-1)^5}=\infty$ and $\lim_{b\to1^+}-\frac{-1}{5(b-1)^5}=\infty$, this integral diverges (to ∞).

2. **Find the second-order Taylor polynomial for $f(x) = \sqrt{x}$ based at $x_0 = 100$. Then use your polynomial to estimate $\sqrt{105}$.**

$$f(x) = x^{1/2} \qquad f(100) = 10$$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \qquad f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$$

$$f''(x) = \frac{-1}{4}x^{-3/2} = \frac{-1}{4x^{3/2}} \qquad f''(100) = \frac{-1}{4 \cdot 100^{3/2}} = \frac{-1}{4000}$$

$$\begin{aligned} P_2(x) &= f(100) + f'(100)(x-100) + \frac{f''(100)}{2!}(x-100)^2 \\ &= 10 + \frac{x-100}{20} - \frac{(x-100)^2}{8000} \end{aligned}$$

Now, $\sqrt{105} \approx P_2(105) = 10 + \dfrac{105-100}{20} - \dfrac{(105-100)^2}{8000} = \dfrac{3279}{320}$.

3. **What is the largest possible error that could have occurred in your estimate of $\sqrt{105}$?**

We know that $|f(x) - P_n(x)| \leq \dfrac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}$.

In this case, $n = 2$, $x_0 = 100$, and $x = 105$.

$K_3 = \text{max of } |f'''(x)| \text{ on } [100, 105] = \text{max of } |\dfrac{3}{8x^{5/2}}| \text{ on } [100, 105] = \dfrac{3}{8 \cdot 100^{5/2}} = \dfrac{3}{800,000}$

Putting this all together, we have $|f(x) - P_2(x)| \leq \dfrac{\frac{3}{800,000}}{3!}|105 - 100|^3 = \dfrac{1}{12800}$.

4. **Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.**

(a) $\displaystyle\int_1^\infty \frac{6 + \cos x}{x^{0.99}}\,dx$

For all $x \geq 1$, we have $\dfrac{6+\cos x}{x^{0.99}} \geq \dfrac{6-1}{x^{0.99}} = \dfrac{5}{x^{0.99}}$ because the minimum value of $\cos x$ is -1.

Since $\displaystyle\int_1^\infty \frac{5\,dx}{x^{0.99}}$ diverges (compute yourself or notice that $p = 0.99 < 1$), we know that the integral in question must diverge too.

(b) $\displaystyle\int_1^\infty \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1}\,dx$

For all $x \geq 1$, we have $\dfrac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \leq \dfrac{4x^3}{x^5} = 4\dfrac{1}{x^2}$. (We've made the denominator smaller and the numerator larger, so the new fraction is larger.)

$$\begin{aligned} 4\int_1^\infty \frac{dx}{x^2} &= 4\lim_{t\to\infty}\int_1^t \frac{dx}{x^2} \\ &= 4\lim_{t\to\infty} \left.\frac{-1}{x}\right|_1^t \\ &= 4\lim_{t\to\infty}\left[\frac{-1}{t} - \frac{-1}{1}\right] \\ &= 4[0 - (-1)] \\ &= 4 \end{aligned}$$

Therefore, the original integral in question must converge to a value less than 4.

5. **The probability density function (pdf) of the length (in minutes) of phone calls on a wireless network is given by $f(x) = ke^{-0.2x}$ where x is the number of minutes. Note that the domain is $x \geq 0$ since we can't have a negative number of minutes.**

   (a) **What must be the value of k?**

   We know that the total area under any pdf must be 1 (because it must account for 100% of events.)

$$\begin{aligned}\int_0^\infty ke^{-0.2x}\,dx &= \lim_{t\to\infty}\int_0^t ke^{-0.2x}\,dx \\ &= \lim_{t\to\infty} \frac{ke^{-0.2x}}{-0.2}\bigg|_0^t \\ &= \lim_{t\to\infty} \frac{ke^{-0.2t}}{-0.2} - \frac{ke^0}{-0.2} \\ &= 0 - \frac{k}{-0.2} \\ &= 5k\end{aligned}$$

   So, we have $5k = 1$ or $k = 0.2$.

   (b) **What fraction of calls last more than 3 minutes?**

$$\begin{aligned}\int_3^\infty 0.2e^{-0.2x}\,dx &= \lim_{t\to\infty}\int_3^t 0.2e^{-0.2x}\,dx \\ &= \lim_{t\to\infty} \frac{0.2e^{-0.2x}}{-0.2}\bigg|_3^t \\ &= \lim_{t\to\infty} -e^{-0.2t} - (-e^{-0.6}) \\ &= 0 + e^{-0.6} \\ &= e^{-0.6} \approx 0.5488\end{aligned}$$

   Note that we could instead have computed $1 - \displaystyle\int_0^3 0.2e^{-0.2x}\,dx$ and gotten the same answer.