On this quiz, **show any RREF'd matrices** you use in any given problem.

1. Let $A = \begin{bmatrix} 7 & 9 & 8 \\ 2 & 1 & 7 \\ 1 & 1 & 2 \\ 5 & 7 & 4 \end{bmatrix}$. Define $T : \mathbb{R}^p \to \mathbb{R}^a$ by $T(\mathbf{v}) = A\mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^p$. Let $\mathbf{m} = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}$

1a. What are the values of p and a? (Label which is which) $p=3 \quad a=4$

1b. Find both $T(\mathbf{m})$ and $T(\mathbf{s})$. If one of them is not possible to find, explain why. Label your answers.

$$T(\vec{m}) = 5\begin{bmatrix} 7 \\ 2 \\ 1 \\ 5 \end{bmatrix} + 8\begin{bmatrix} 9 \\ 1 \\ 1 \\ 7 \end{bmatrix} + 3\begin{bmatrix} 8 \\ 7 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 131 \\ 39 \\ 19 \\ 93 \end{bmatrix}$$

$T(\vec{s})$ cannot be computed because there are only $\underline{3}$ column vectors in A yet $\vec{s}$ supplies $\underline{4}$ weights. (The "extra" weight means we don't know how to make a LC of 3 vectors by using 4 weights).

1c. Find all $\mathbf{x}$ for which $T(\mathbf{x}) = \begin{bmatrix} 20 \\ -10 \\ 0 \\ 20 \end{bmatrix}$. This is the same question as "Find all solutions of $A\vec{x} = \begin{bmatrix} 20 \\ -10 \\ 0 \\ 20 \end{bmatrix}$"

which we answer using $\operatorname{rref}\left(\left[A \,\middle|\, \begin{smallmatrix} 20 \\ -10 \\ 0 \\ 20 \end{smallmatrix}\right]\right) = \operatorname{rref}\left(\left[\begin{array}{ccc|c} 7 & 9 & 8 & 20 \\ 2 & 1 & 7 & -10 \\ 1 & 1 & 2 & 0 \\ 5 & 7 & 4 & 20 \end{array}\right]\right) = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 5 & -10 \\ 0 & 1 & -3 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Longrightarrow \begin{cases} x_1 = -10 - 5x_3 \\ x_2 = 10 + 3x_3 \\ x_3 \text{ is free} \end{cases}$, where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(which, in parametric vector form, is $\vec{x} = \begin{bmatrix} -10 \\ 10 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}$ where x_3 is free)

1e. Today is Friday the 13th, so the vector $\mathbf{u} = \begin{bmatrix} 13 \\ 13 \\ 13 \\ 13 \end{bmatrix}$ is extremely unlucky. Explain why $\mathbf{u}$ is not in the image of T.

We're being told to show that there is no vector $\vec{v}$ in $\mathbb{R}^3$ for which $T(\vec{v}) = \vec{u}$.
Let's $\operatorname{rref}(A|\vec{u})$ and find out: we get $\operatorname{rref}(A|\vec{u}) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$, the third row of which represents $0x_1 + 0x_2 + 0x_3 = 1$. Since this equation has no solutions, the equation $A\vec{v} = \vec{u}$ is inconsistent, and so $T(\vec{v}) = \vec{u}$ also has no solutions, which indeed says $\vec{u}$ is NOT in the image of T

2. Suppose that T is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. There are two conditions that T must satisfy in order to be a *linear* transformation. They are:

2a. the equation $\boxed{T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})}$ is true for every pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, and

2b. the equation $\boxed{T(\alpha\vec{w}) = \alpha\, T(\vec{w})}$ is true for any scalar $\alpha \in \mathbb{R}$ and vector $\mathbf{w} \in \mathbb{R}^n$.

2c. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T\left(\begin{bmatrix} s \\ t \end{bmatrix}\right) = \begin{bmatrix} st \\ 3s + 2t + 6 \end{bmatrix}$. Show that T fails to satisfy the equation in (2a) using the vectors $\mathbf{u} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

We find $T(\vec{u}+\vec{v}) = T\left(\begin{bmatrix} 8 \\ 7 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix}\right) = T\left(\begin{bmatrix} 12 \\ 12 \end{bmatrix}\right) = \begin{bmatrix} 12 \cdot 12 \\ 3 \cdot 12 + 2 \cdot 12 + 6 \end{bmatrix} = \begin{bmatrix} 144 \\ 66 \end{bmatrix}$

and $T(\vec{u}) + T(\vec{v}) = \begin{bmatrix} 8 \cdot 7 \\ 3 \cdot 8 + 2 \cdot 7 + 6 \end{bmatrix} + \begin{bmatrix} 4 \cdot 5 \\ 3 \cdot 4 + 2 \cdot 5 + 6 \end{bmatrix} = \begin{bmatrix} 56 \\ 44 \end{bmatrix} + \begin{bmatrix} 20 \\ 28 \end{bmatrix} = \begin{bmatrix} 76 \\ 72 \end{bmatrix}$

Since $\begin{bmatrix} 144 \\ 66 \end{bmatrix} \neq \begin{bmatrix} 76 \\ 72 \end{bmatrix}$, T fails condition in 2a.