MATH106A CALCULUS II - PROF. P. WONG

EXAM I - FEBRUARY 6, 2015

NAME:

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

Advice: DON'T spend too much time on a single problem.

Problems	**Maximum Score**	**Your Score**
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
Total	100	

1.(10 pts.)(a) Evaluate the indefinite integral (be sure to show all your work)

$$\int \frac{4z\ dz}{\sqrt{2z^2+1}}.$$

Let $u = 2z^2 + 1$**. Then** $du = 4z\ dz$**. It follows that**

$$\begin{aligned}\int \frac{4z\ dz}{\sqrt{2z^2+1}} &= \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\ du \\ &= \frac{u^{1/2}}{1/2} + C \\ &= 2\sqrt{2z^2+1} + C.\end{aligned}$$

(10 pts.) (b) Find the **exact value** of the definite integral (be sure to show all your work)

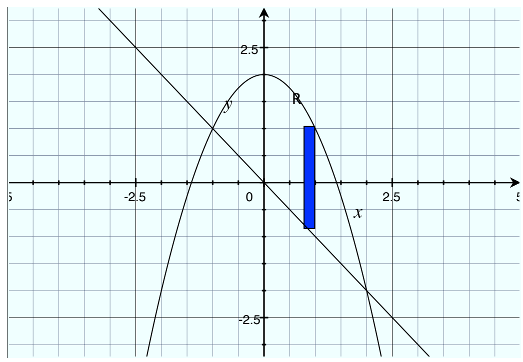
$$\int_0^{\pi} 3\cos^2 x \sin x\ dx.$$

Let $u = \cos x$ **so that** $du = -\sin x\ dx$**. When** $x = 0, u = \cos 0 = 1$**. When** $x = \pi, u = \cos\pi = -1$**. Thus,**

$$\begin{aligned}\int_0^{\pi} 3\cos^2 x \sin x\ dx &= \int_1^{-1} 3u^2(-du) \\ &= \int_{-1}^{1} 3u^2\ du = u^3\Big|_{-1}^{1} \\ &= 1 - (-1) = 2.\end{aligned}$$

2. (12 pts.) (a) Consider the region A bounded by the curve $y = 2 - x^2$ and the line $y = -x$. Find the **exact area** of the region A.

The curve $y = 2 - x^2$ and the line $y = -x$ intersect when $2 - x^2 = -x$ or $x^2 - x - 2 = 0$. It follows that $x = -1$ or $x = 2$. Using vertical (blue) slices, the area of A is given



by

$$\begin{aligned}\text{area}(A) &= \int_{-1}^{2} (2 - x^2) - (-x)\ dx \\ &= \int_{-1}^{2} 2 + x - xr\ dx \\ &= 2x - \frac{x^2}{2} - \frac{x^3}{3}\Big|_{-1}^{2} \\ &= \frac{9}{2}.\end{aligned}$$

(8 pts.)(b) Find the (**exact**) length of the curve $y = \frac{1}{3}(x^2 + 2)^{3/2}$ between $x = 0$ and $x = 3$.

The length of the curve in question is given by the following integral

$$L = \int_0^3 \sqrt{1 + (f'(x))^2}\ dx.$$

Since $f(x) = \frac{1}{3}(x^2 + 2)^{3/2}$, it follows that $f'(x) = \frac{1}{3} \cdot \frac{3}{2}(x^2 + 2)^{1/2} \cdot (2x) = x(x^2 + 2)^{1/2}$. Now,

$$\begin{aligned}L &= \int_0^3 \sqrt{1 + [x(x^2 + 2)^{1/2}]^2}\ dx \\ &= \int_0^3 \sqrt{1 + x^2(x^2 + 2)}\ dx = \int_0^3 \sqrt{1 + 2x^2 + x^4}\ dx \\ &= \int_0^3 \sqrt{(1 + x^2)^2}\ dx = \int_0^3 1 + x^2\ dx \\ &= \left(x + \frac{x^3}{3}\right)\Big|_0^3 = 3 + 9 = 12.\end{aligned}$$

3. (12 pts.) Consider a function h on the interval $[1, 4]$.

x	1	1.5	2	2.5	3	3.5	4
$h(x)$	-1	1	3	2	0	-3	-2

Find T_6, M_3 using the trapezoid rule and the mid-point rule respectively for estimating the definite integral $\int_1^4 h(x)\, dx$.

Consider the Left Hand Sum $L_6 = [h(1) + h(1.5) + h(2) + h(2.5) + h(3) + h(3.5)] \cdot \Delta x$ **where** $\Delta x = 0.5$**. It follows that** $L_6 = [(-1) + (1) + (3) + (2) + (0) + (-3)](0.5) = 1$**. Simialrly, the Right Hand Sum** $R_6 = [h(1.5)+h(2)+h(2.5)+h(3)+h(3.5)+h(4)] \cdot \Delta x = [(1) + (3) + (2) + (0) + (-3) + (-2)](0.5) = 0.5$**. Now,**

$$T_6 = \frac{L_6 + R_6}{2} = \frac{1 + 0.5}{2} = 0.75.$$

For the Midpoint Rule, $M_3 = [h(1.5) + h(2.5) + h(3.5)] \cdot \Delta x$ **where** $\Delta x = 1$**. Thus,**

$$M_3 = [(1) + (2) + (-3)](1) = 0.$$

(8 pts.)(b) Recall that the error committed by using the left hand sum approximation L_n is less than or equal to $\frac{K_1 \cdot (b-a)^2}{2n}$ where $|f'(x)| \leq K_1$ for some constant K_1 over the interval $[a, b]$. Use this result to give an upper bound for the error committed by L_{10} for

$$I = \int_0^1 e^{-t^2}\, dt.$$

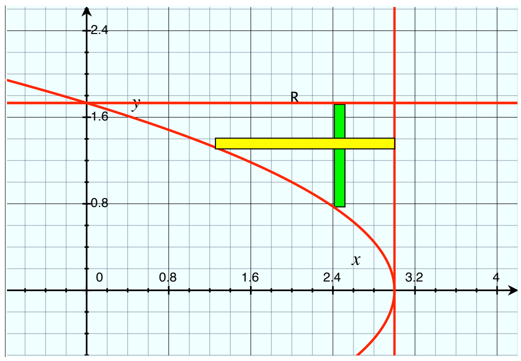
Here, $f(t) = e^{-t^2}$ **so** $f'(t) = (-2t)e^{-t^2}$ **and** $|f'(t)| = 2|t|e^{-t^2}$**. For** $0 \leq t \leq 1$**,** $|t| \leq 1$ **and** $e^{-t^2} \leq e^0 = 1$**. Thus, we can choose** $K_1 = 2 \cdot 1 \cdot 1 = 2$**. It follows that the error committed by** L_{10} **is no greater than** $\frac{K_1 \cdot (1-0)^2}{2(10)} = \frac{2}{20} = 0.1$.

Alternatively, one can estimate the maximum value of $2te^{-t^2}$ **over the the interval** $[0, 1]$ **and use that value as** K_1**.**

4. Let R be the region in the first quadrant bounded by the curve $x = 3 - y^2$, the line $y = \sqrt{3}$, and the line $x = 3$.

(12 pts.) (a) Find the **exact volume** of the solid obtained from rotating the region R around the y-axis. [Hint: sketch a picture of the region R first.]

Slice the region R horizontally and then rotate around the y-axis.



Thus a typical slice (yellow) looks like a "washer" with thickness Δy ranging from (bottom slice) $y = 0$ to (top slice) $y = \sqrt{3}$. The larger radius (x-coordinate) is always 3 and the smaller radius is simply $x = 3 - y^2$. Thus the volume of a typical slice is approximately $[\pi(3)^2 - \pi(3 - y^2)^2] \cdot \Delta y$. It follows that the volme of the solid of revolution is given by

$$V = \int_0^{\sqrt{3}} \pi[3^2 - (3 - y^2)^2] \; dy = \pi \int_0^{\sqrt{3}} [9 - (9 - 6y^2 + y^4)] \; dy$$

$$\pi \int_0^{\sqrt{3}} 6y^2 - y^4 \; dy = \pi \left(2y^3 - \frac{y^5}{5}\right)\Bigg|_0^{\sqrt{3}} = \frac{21\sqrt{3}\pi}{5}.$$

(8 pts.) (b) Set up (do not evaluate) a definite integral representing the volume of the solid obtained from rotating the region R around the x-axis.

If we use vertical slices (green) each of which looks like a "washer" then the volume of revolution (about the x-axis) is given by

$$V = \int_0^3 \pi(\sqrt{3})^2 - \pi(\sqrt{3 - x})^2 \; dx = \pi \int_0^3 (3 - (3 - x)) \; dx = \pi \frac{x^2}{2}\Bigg|_0^3 = \frac{9\pi}{2}.$$

If we use horizontal slices (rotate the yellow slices around x-axis) then each one is a "cylindrical shell". Then the volume of revolution is given by

$$V = \int_0^{\sqrt{3}} 2\pi y(3 - (3 - y^2)) \; dy = 2\pi \int_0^{\sqrt{3}} y^3 \; dy = 2\pi \frac{y^4}{4}\Bigg|_0^{\sqrt{3}} = \frac{9\pi}{2}.$$

5. (10 pts.)(a) Consider the initial value problem

$$\frac{dy}{dx} = 2xe^{-y}$$

with $y(0) = 0$. Use the technique of separation of variables to solve the Initial Value Problem.

By separating the variables, we get $e^y\ dy = 2x\ dx$ or $\int e^y\ dy = \int 2x\ dx$. It follows that

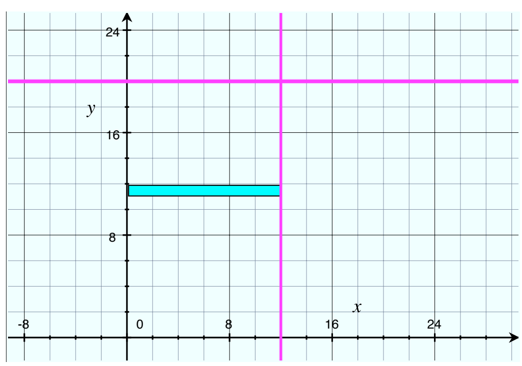
$$e^y = x^2 + C. \tag{1}$$

Since $y(0) = 0$, equation (1) becomes $e^0 = 0 + C$ or $C = 1$. Thus,

$$e^y = x^2 + 1 \qquad \textbf{or} \qquad y = \ln(x^2 + 1).$$

(10 pts.)(b) A rectangular tank (width 10 ft, length 12 ft, height 20 ft), with its top at ground level, is used to catch runoff water. Assume that the water weighs 62.4 lb/ft^3. How much work does it take to empty the tank by pumping the water back to ground level once the tank is full? [Draw a picture!]

Put a coordinate system with the y-axis measuring the depth of the water. Let the x-axis be parallel to the bottom face of the tank and be placed 20 feet below the surface. A thin (light blue) layer of water at height y from the bottom of the tank has $(20 - y)$ feet to be lifted to the surface.



Moreover, the thin layer of water has volume approximately $(12 \times 10) \cdot \Delta y$. Thus the total work required is given by the integral

$$\begin{aligned} W &= \int_0^{20} (62.4)(12 \times 10)(20 - y)\ dy \\ &= (62.4)(120) \int_0^{20} (20 - y)\ dy \\ &= (62.4)(120) \left(20y - \frac{y^2}{2}\right)\Bigg|_0^{20} \qquad = 1497600\text{ft} - \text{lb}. \end{aligned}$$