Math 106 Winter 2015

Test 1 (50 points)

Name: Solutions

Show all your work to receive full credit for a problem. Points will be taken off if you do not show how you arrived at your answer, even if the final answer is correct.

Please keep your written answers brief; be clear and to the point. Points will be taken off for rambling and for incorrect or irrelevant statements and for multiple solutions to the same problem.

Do not use the calculator integral function. Whenever possible, find the exact values of integrals by finding antiderivatives or using the table of integrals.

When you use a formula from the table of integrals, mention the formula number and the value(s) of any constant(s) that you may need.

Give exact answers. If needed, round off your answers to four decimal places.

Include units in your answers wherever possible.

There are six questions. Questions are printed on both sides of a page.

You may use any of the following facts:

Arclength$=\int_a^b \sqrt{1+(f'(x))^2}\,dx$

$|I-L_n| \le \dfrac{K_1(b-a)^2}{2n}$ $\qquad$ $|I-R_n| \le \dfrac{K_1(b-a)^2}{2n}$

$|I-T_n| \le \dfrac{K_2(b-a)^3}{12n^2}$ $\qquad$ $|I-M_n| \le \dfrac{K_2(b-a)^3}{24n^2}$

Below are product rule, quotient rule and chain rule for derivatives.

$(uv)' = u'v+uv'$ $\qquad$ $\left(\dfrac{u}{v}\right)' = \dfrac{vu'-uv'}{v^2}$ $\qquad$ $(f\circ g)'(x) = f'(g(x))\,g'(x)$

1. (9 points) Evaluate the following integral exactly. (You may use formulas 1-18 only from the table of integrals for this problem.)

$$\int_{-1}^{2} 5x\sqrt{3-x}\,dx.$$

First we will find $\int 5x\sqrt{3-x}\,dx$.

Let $u = 3-x$, $du = -dx$. So $dx = -du$.

Also $u = 3-x$ gives $x = 3-u$.

So $\int 5x\sqrt{3-x}\,dx = \int 5(3-u)\sqrt{u}\,(-du) = -5\int (3-u)\,u^{1/2}\,du$

$$= -5\int \left(3u^{1/2} - u^{3/2}\right) du$$

$$= -5\left[3\frac{u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right]$$

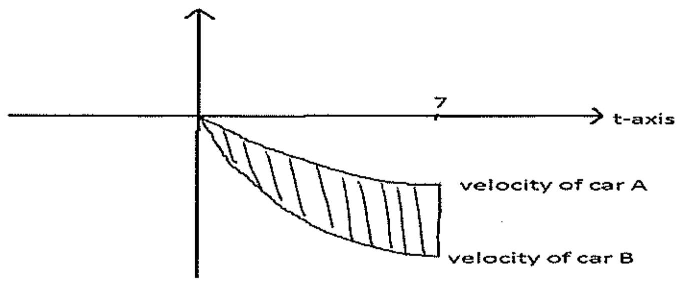
$$= -5\left[2(3-x)^{3/2} - \frac{2}{5}(3-x)^{5/2}\right] + C.$$

Then $\int_{-1}^{2} 5x\sqrt{3-x}\,dx = -5\left[2(3-x)^{3/2} - \frac{2}{5}(3-x)^{5/2}\right]_{-1}^{2}$

$$= -5\left[\left(2 - \frac{2}{5}\right) - \left(2\cdot 4^{3/2} - \frac{2}{5}\cdot 4^{5/2}\right)\right]$$

$$= -5\left[2 - \frac{2}{5} - 16 + \frac{64}{5}\right] = 8.$$

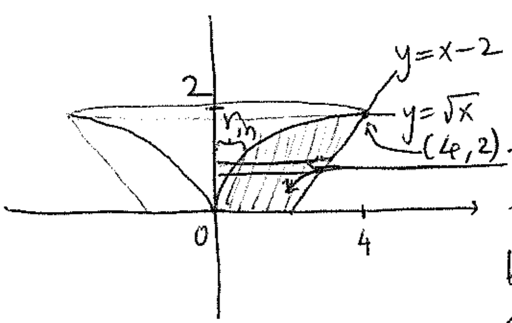
2. (5 points) Suppose $v_A(t)$ and $v_B(t)$ indicate the eastward velocities of cars A and B respectively in miles per hour. The graphs of the two functions are given below. Shade the area bounded by the two curves and the line $t = 7$ and write (but do not evaluate) an integral to find this area. What does this integral represent in car talk?



Shaded area $= \int_0^7 (V_A(t) - V_B(t))\,dt$

The integral gives the distance in miles between the two cars at the end of seven hours.

3. (9 points) Sketch the region bounded by the curves $y = \sqrt{x}$, $y = x - 2$ and the x–axis. Write (but do not evaluate) an integral to find the volume of the solid obtained by rotating the region about the y–axis. Clearly show the steps used in writing down the integral. For example, write down the shape of the slice and the cross-sectional area of the slice.



This is the region bounded by the two curves and the x-axis.

Slice is a washer.

Cross-sectional area of slice $= \pi r_{out}^2 - \pi r_{in}^2$.

$r_{in} = x = y^2$ ($y = \sqrt{x}$. So $x = y^2$).

$r_{out} = x = y+2$ ($y = x-2$. So $x = y+2$).

~~Vol~~ Volume of slice $= \pi\left((y+2)^2 - (y^2)^2\right)dy$.

Volume of solid $= \int_0^2 \pi\left((y+2)^2 - y^4\right)dy$.

4. (9 points)

(a) Write (but do not evaluate) an integral to find the exact length of the curve $y = \ln x$ from $x = 1$ to $x = 2$.

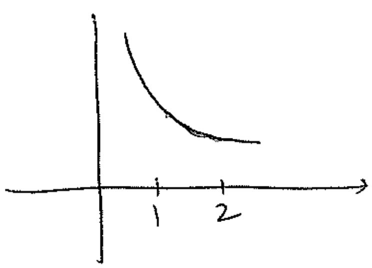
$$y = \ln x$$

$$y' = \frac{1}{x}$$

$$\text{Arclength} = \int_1^2 \sqrt{1+(y')^2}\,dx$$

$$= \int_1^2 \sqrt{1+\frac{1}{x^2}}\,dx$$

(b) If you use M_{25} to approximate the integral you wrote in part (a), does it underestimate or overestimate the exact integral value? Justify your answer. Do not evaluate the integral or find M_{25} to answer this question.



Graph of $\sqrt{1+\frac{1}{x^2}}$

On $[1,2]$, the function $\sqrt{1+\frac{1}{x^2}}$ is concave up as seen in the graph.

So M_{25} underestimates the integral

$$\int_1^2 \sqrt{1+\frac{1}{x^2}}\,dx.$$

5. (9 points) Let $I = \int_{0.5}^{3.5} f(x)\,dx$ where $f(x)$ is a function with the following properties:

$f(x) \geq 0$, $-5 \leq f'(x) \leq 2$ and $-2 \leq f''(x) \leq 1$ for all x in the interval $[0.5, 3.5]$.

What is the least value of n which guarantees that R_n approximates I within ± 0.2? Justify your answer.

From the error bound theorem, we know

$$|I - R_n| \leq \frac{K_1 (b-a)^2}{2n}.$$

We want to find the least value of n such that

$$|I - R_n| \leq 0.2.$$

So if we solve $\frac{K_1 (b-a)^2}{2n} \leq 0.2$, we will get the value of n we want.

$a = 0.5$, $b = 3.5$.

Since $-5 \leq f'(x) \leq 2$, $|f'(x)| \leq 5$ on $[0.5, 3.5]$.

Hence $K_1 = 5$.

So $\frac{5(3.5-0.5)^2}{2n} \leq 0.2$.

$\frac{45}{0.4} \leq n$. i.e $n \geq 112.5$

Since n has to be a positive integer as it represents the number of subintervals used in the approximation, the least value of n is 113.

6. (9 points) Solve the DE: $y'e^{2x} + y' = y^3 e^x$. (You may use formulas 1-18 only from the table of integrals for this problem.)

$$y'(e^{2x}+1) = y^3 e^x.$$

$$\frac{dy}{dx}(e^{2x}+1) = y^3 e^x.$$

So $$\frac{1}{y^3}\,dy = \frac{e^x}{e^{2x}+1}\,dx.$$

$$\int \frac{1}{y^3}\,dy = \int \frac{e^x}{e^{2x}+1}\,dx.$$

$$\int \frac{1}{y^3}\,dy = \int y^{-3}\,dy = \frac{y^{-2}}{-2} = -\frac{1}{2y^2}.$$

For $\int \frac{e^x}{e^{2x}+1}\,dx$, let $u = e^x$. Then $du = e^x\,dx$.

$$\int \frac{e^x}{(e^x)^2+1}\,dx = \int \frac{du}{u^2+1} = \arctan(u) \quad \text{(Formula 13 with } a=1\text{)}.$$

$$= \arctan(e^x).$$

Thus, $$\frac{-1}{2y^2} = \arctan(e^x) + C.$$

$$\frac{1}{y^2} = -2(\arctan(e^x)+C).$$

$$y^2 = \frac{1}{-2(\arctan(e^x)+C)}$$

So $$y = \sqrt{\frac{1}{-2(\arctan(e^x)+C)}}.$$