Math 106: Review for Exam I - SOLUTIONS

1. **Find the following.** [Substitution tip: usually let u = a function that's "inside" another function, especially if du (possibly off by a multiplying constant) is also present in the integrand.]

   (a) Let $u = \sqrt{x}$, so $du = \dfrac{dx}{2\sqrt{x}}$ and $2\,du = \dfrac{dx}{\sqrt{x}}$.

   Now we'll change the limits.

   If $x = 1$, then $u = \sqrt{1} = 1$ and if $x = 4$, then $u = \sqrt{4} = 2$.

$$\begin{aligned}\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx &= \int_1^2 e^u \cdot 2\,du \\ &= 2e^u\Big|_1^2 \\ &= 2e^2 - 2e\ (\approx 9.342)\end{aligned}$$

   (b) Let $u = \cos(5x)$, so $du = -5\sin(5x)\,dx$ and $-\dfrac{du}{5} = \sin(5x)\,dx$.

   Now we'll change the limits.

   If $x = \pi$, then $u = \cos(5 \cdot \pi) = -1$ and if $x = 2$, then $u = \cos(5 \cdot 2\pi) = 1$.

$$\begin{aligned}\int_\pi^{2\pi} \cos^7(5x)\sin(5x)\,dx &= \int_{-1}^1 u^7 \cdot \frac{-du}{5} \\ &= -\frac{1}{5}\int_{-1}^1 u^7\ du \\ &= -\frac{1}{5}\frac{u^8}{8}\Big|_{-1}^1 \\ &= -\frac{1}{40}(1^8 - (-1)^8) \\ &= 0\end{aligned}$$

   (c) Use $u = x^3$, so $du = 3x^2\ dx$ and $\dfrac{du}{3} = x^2\ dx$.

$$\begin{aligned}\int \frac{7x^2}{1+x^6}\,dx &= 7\int \frac{\frac{du}{3}}{1+u^2} \\ &= \frac{7}{3}\arctan u + C \\ &= \frac{7}{3}\arctan(x^3) + C\end{aligned}$$

(d) Use $u = 10 - x$, so $du = -dx$ and $dx = -du$.

$$\begin{aligned}\int \boldsymbol{x\sqrt{10-x}\,dx} &= \int (10-u)\sqrt{u}(-du) \qquad \text{Since } u = 10-x, \text{ we know } x = 10-u.\\ &= \int (u-10)\sqrt{u}\,du\\ &= \int (u^{3/2} - 10u^{1/2})du\\ &= \frac{2}{5}u^{5/2} - \frac{20}{3}u^{3/2} + C\\ &= \frac{2}{5}(10-x)^{5/2} - \frac{20}{3}(10-x)^{3/2} + C\end{aligned}$$

2. **If $f(x)$ is decreasing and concave up, put the following quantities in ascending order.**

L_{100}, R_{100}, T_{100}, M_{100}, $\int_a^b f(x)\,dx$

$$R_{100} < M_{100} < \int_a^b f(x)\,dx < T_{100} < L_{100}$$

What can you say with certainty about where S_{200} would fit into your list above?

It would be somewhere between M_{100} and T_{100} but we don't know how it compares to $\int_a^b f(x)\,dx$.

3. **Find the best possible left, right, midpoint, trapezoidal, and Simpson's approximations to $\int_4^{12} f(t)\,dt$ given the data in the table below.**

t	4	6	8	10	12
$f(t)$	15	11	8	4	3

$L_4 = (15 + 11 + 8 + 4)(2) = 76$ $\qquad$ $R_4 = (11 + 8 + 4 + 3)(2) = 52$ $\qquad$ $T_4 = \dfrac{L_4 + R_4}{2} = 64$

We cannot compute M_4 because it requires the values of f at $x = 5$, 7, 9, and 11. Instead, we do M_2.
$M_2 = (11 + 4)(4) = 60$

Now, to find S_4, we need $T_2 = \dfrac{L_2 + R_2}{2} = \dfrac{(15+8)(4) + (8+3)(4)}{2} = 68$.

$$S_4 = \frac{2M_2 + T_2}{3} = \frac{2(60) + 68}{3} = \frac{188}{3} = 62.\overline{6}$$

4. **Find bounds for each of the following errors if $I = \int_2^7 \ln x\,dx$.**

(a) $|I - L_{100}| \leq \dfrac{K_1(b-a)^2}{2n} = \dfrac{\frac{1}{2}(7-2)^2}{2(100)} = \dfrac{1}{16}$

$K_1 = \text{max of } |f'(x)| \text{ on } [2,7] = \text{max of } \dfrac{1}{x} \text{ on } [2,7] = \dfrac{1}{2}$ (occurs at $x = 2$)

(b) $|I - T_{100}| \leq \dfrac{K_2(b-a)^3}{12n^2} = \dfrac{\frac{1}{4}(7-2)^3}{12(100)^2} = \dfrac{1}{3840}$

$K_2 = \text{max of } |f''(x)| \text{ on } [2,7] = \text{max of } \dfrac{1}{x^2} \text{ on } [2,7] = \dfrac{1}{4}$ (occurs at $x = 2$)

(c) $|I - M_{100}| \le \dfrac{K_2(b-a)^3}{24n^2} = \dfrac{\frac{1}{4}(7-2)^3}{24(100)^2} = \dfrac{1}{7680}$

K_2 = same as in previous part

5. **If $I = \int_2^7 \ln x \, dx$, how many subdivisions are required to obtain a trapezoidal sum approximation with error of at most 1/1,000,000?**

From part (b) above, we know that $|I - T_n| \le \dfrac{K_2(b-a)^3}{12n^2} = \dfrac{\frac{1}{4}(7-2)^3}{12n^2} = \dfrac{125}{48n^2}$.

Thus, we want $\dfrac{125}{48n^2} \le \dfrac{1}{1,000,000}$.

Multiplying each side by $1,000,000n^2$ gives $\dfrac{125,000,000}{48} \le n^2$.

Taking the square root of each side results in $\sqrt{\dfrac{125,000,000}{48}} \le n$.

Since $\sqrt{\dfrac{125,000,000}{48}} = 1613.743...$, we must at least 1614 subdivisions.

6. **Solve the differential equation $dy/dx = 2xy + 6x$ if the solution passes through $(0, 5)$.**

$$\frac{dy}{dx} = 2xy + 6x$$

$$\frac{dy}{dx} = 2x(y+3)$$

$$\frac{dy}{y+3} = 2x \, dx \qquad \text{Separate the variables.}$$

$$\int \frac{dy}{y+3} = \int 2x \, dx$$

$$\ln|y+3| = x^2 + C$$

$$|y+3| = e^{x^2+C} \qquad \text{Exponentiate each side to remove the ln.}$$

$$y + 3 = \pm e^C e^{x^2} \qquad |w| = z \text{ means } w = \pm z.$$

$$y = -3 + Ae^{x^2} \qquad \text{Replace } \pm e^C \text{ with } A.$$

Now we use the initial condition $y(0) = 5$ to find the value of A.

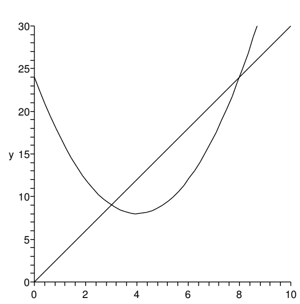
We have $5 = -3 + Ae^0 \Rightarrow A = 8$, so the solution is $y = -3 + 8e^{x^2}$.

7. **Write integrals equal to**

(a) **the arc length of $y = x^2$ on the interval $[1, 5]$**

arc length of $y = f(x)$ on $[a, b] = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx = \int_1^5 \sqrt{1 + (2x)^2} \, dx$ (≈ 24.395)

(b) **the area bounded by $y = x^2 - 8x + 24$ and $y = 3x$**



First, find where the curves intersect.

$$\begin{aligned} x^2 - 8x + 24 &= 3x \\ x^2 - 11x + 24 &= 0 \\ (x-3)(x-8) &= 0 \\ \Rightarrow\ & x = 3, x = 8 \end{aligned}$$

Between $x = 3$ and $x = 8$, $y = 3x$ is above $y = x^2 - 8x + 24$. (Plug in $x = 5$ or graph to check.)

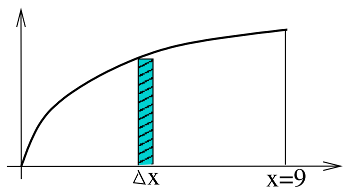
So, the area between them is

$$\int_3^8 [3x - (x^2 - 8x + 24)]\, dx.$$

[This equals 125/6.]

8. Consider the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$. Write an integral equal to the volume generated if this region is revolved about

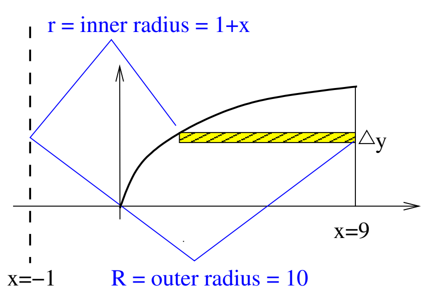
(a) **the x-axis**



$$\begin{aligned} \text{volume of slice } &\approx \pi r^2 \Delta x \\ &= \pi y^2 \Delta x \\ &= \pi(\sqrt{x})^2 \Delta x \\ &= \pi x \Delta x \end{aligned}$$

$$\text{total volume} = \pi \int_0^9 x\, dx$$

(b) **the line $x = -1$**



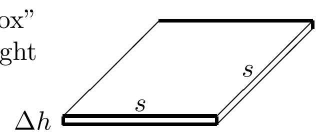
$$\begin{aligned} \text{volume of slice } &\approx \pi R^2 \Delta y - \pi r^2 \Delta y \\ &= \pi(10^2)\Delta y - \pi(1+x)^2 \Delta y \\ &= \pi[100 - (1+y^2)^2]\Delta y \end{aligned}$$

$$\text{total volume} = \pi \int_0^3 [100 - (1+y^2)^2]\, dy$$

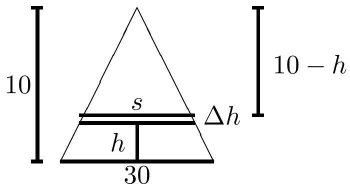
9. A pyramid has a square base 30 feet to a side and a height of 10 feet. Write integrals equal to

(a) **the volume of the pyramid**

We slice horizontally, so each slice is a "box" with a square top and bottom and a height (thickness) of Δh,



The picture shown below is a vertical cross-section through the center of the pyramid.



Similar triangles: $\frac{10}{30} = \frac{10-h}{s} \Rightarrow s = 3(10-h)$.

volume of slice $\approx s^2\Delta h \approx [3(10-h)]^2\Delta h$

total volume $= \int_0^{10} [3(10-h)]^2\, dh$

(b) **the work done in pumping all the fluid to a point 5 feet above the pyramid if the pyramid is filled to a height of 8 feet with water (which weighs 62.4 pounds per cubic foot) [Students in the 1:10 section should omit this part.]**

We use the same sketch as in the previous part.

$$\begin{aligned}
\text{volume of slice} &\approx s^2\Delta h \approx [3(10-h)]^2\Delta h && \text{From above.}\\
\text{weight of slice} &\approx 62.4[3(10-h)]^2\Delta h && \text{Weight=(density)(volume).}\\
\text{work to lift slice} &\approx 62.4[3(10-h)]^2\Delta h(15-h) && \text{Work=(force)(distance); here, force=weight.}\\
\text{total work} &= 62.4\int_0^8 [3(10-h)]^2(15-h)\, dh
\end{aligned}$$