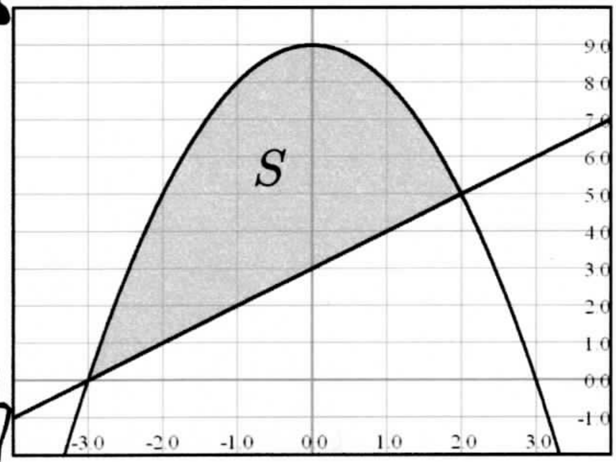
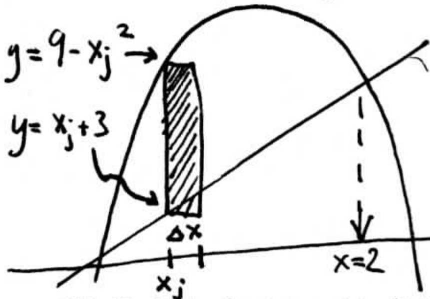
SUGGESTED SOLUTIONS



1. Consider the region S in the plane of all points which are between the graphs of the parabola $y = 9 - x^2$ and the straight line $y = x + 3$. The region S is shown in the figure to the right.

1A. Set up the integral which represents the area of S if the corresponding approximations are rectangles each of whose base width is $\triangle x$ and each rectangle goes from a top curve down to a bottom curve. That is, the integral is of the form $\int dx$.



height of [] is $(9-x_j^2)-(x_j+3)$
(top curve - bottom curve)

$$\therefore \boxed{\int_{x=-3}^{x=2} \left((9-x^2)-(x+3)\right) dx}$$

1B. Evaluate the integral in (1A) to find that area. Show all your work.

we can simplify the $\int$ a little:

$$\int_{x=-3}^{x=2} (6 - x^2 - x)\, dx$$

$$= 6x - \frac{x^3}{3} - \frac{x^2}{2} \Big|_{-3}^{2}$$

$$= \left(12 - \frac{8}{3} - \frac{4}{2}\right) - \left(-18 + \frac{27}{3} - \frac{9}{2}\right)$$

$$= 12 - 2 + 18 - 9 - 8/3 + 9/2$$

$$= 19 - 8/3 + 9/2 = \frac{19\cdot 6 - 8\cdot 2 + 9\cdot 3}{6} = \frac{114-16+27}{6} = \frac{125}{6} = 20\tfrac{5}{6}$$

alternatively:

$$(12 - 8/3 - 4/2) - (-18 + \tfrac{27}{3} - 9/2)$$

$$= (7.3\overline{3}) - (-13.5)$$

$$= 20.83\overline{3}$$

2A. Set up the integral which gives the arc length of the curved part of the boundary of S, that is, the arc length of the graph of $y = 9 - x^2$ from $x = -3$ to $x = 2$.

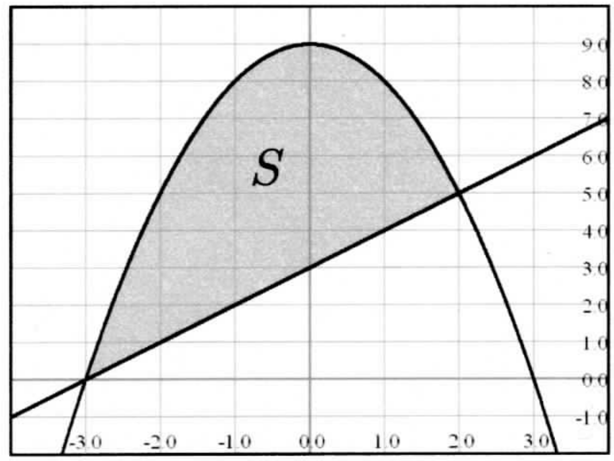
$$\int_{-3}^{2} \sqrt{(f'(x))^2 + 1}\, dx \quad \text{where } f(x) = 9 - x^2 \text{ becomes}$$

$$\int_{-3}^{2} \sqrt{(-2x)^2 + 1}\, dx = \int_{-3}^{2} \sqrt{4x^2+1}\, dx$$

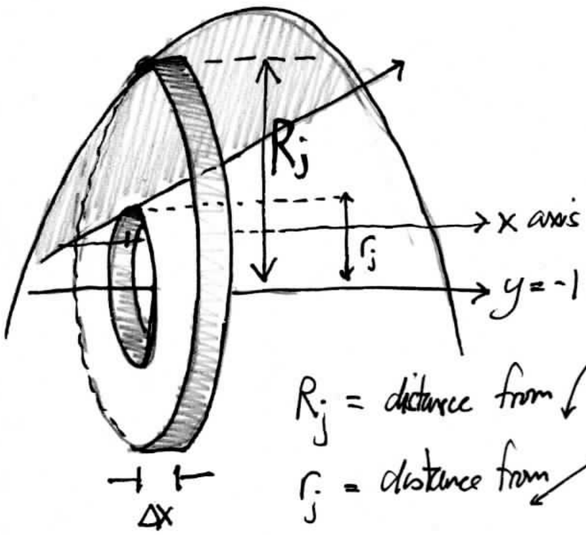
2B. The integral in 2A is "doable" — the back of your book has a formula for the antiderivative you'd need. But it's so complicated, and in practice, a good numerical approximation will do. Indeed, find the MID(50) approximation for the integral in 2A.

MID(50) yields $\boxed{14.39224\ldots}$

3. Again, consider the region S from problem 1, of all points which are between the graphs of the parabola $y = 9 - x^2$ and the straight line $y = x + 3$. The region S is shown in the figure to the right.



3A. Set up the integral(s) which represents the volume of the solid of revolution obtained by revolving S around the line $y = -1$. (Do not evaluate the integral(s)).



R_j = distance from ∩ to $y=-1$ = top − bottom = $(9 - x_j^2) - {-1} = 10 - x_j$

r_j = distance from ↗ to $y=-1$ = top − bottom = $(x_j + 3) - {-1} = x_j + 4$

∴ the volume of that "washer" is $\pi R_j^2 - \pi r_j^2 = \pi(10 - x_j)^2 \Delta x - \pi(x_j + 4)^2 \Delta x$

and so the corresponding integral is $\displaystyle\int_{-3}^{2} \left(\pi(10 - x)^2 - \pi(x+4)^2\right) dx$

3B. Now set up the integral(s) which represents the volume of the solid of revolution obtained by revolving S around the line $x = -3$. (Do not evaluate the integral(s)).

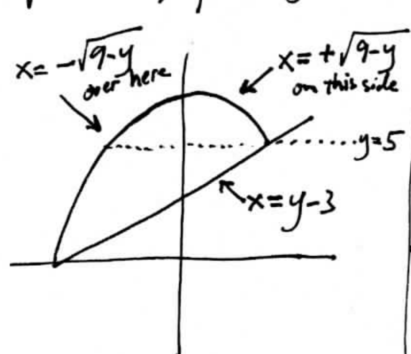
This will require TWO integrals, as the formula for the outer radius R_j changes at $y = 5$

We need to describe the curves involved as functions of y. SO:

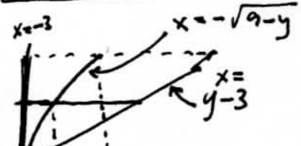
If $y = 9 - x^2$ then $x^2 = 9 - y$ so DONT FORGET $x = \pm\sqrt{9-y}$

and if $y = x + 3$ then $x = y - 3$

Proper labeling of the figure is:



from 0 to 5: y_j

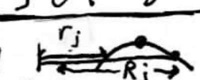


R_j = right − left $= (y_j - 3) - {-3} = y_j - 3 + 3 = y_j$

r_j = right − left $= -\sqrt{9-y_j} - {-3} = 3 - \sqrt{9 - y_j}$

resulting in $\displaystyle\int_{y=0}^{y=5} \left(\pi y^2 - \pi\left(3 - \sqrt{9-y}\right)^2\right) dy.$

from 5 to 9



$r_j = -\sqrt{9-y_j} - {-3} = 3 - \sqrt{9-y_j}$

$R_j = +\sqrt{9-y_j} - {-3} = 3 + \sqrt{9-y_j}$

resulting in $\displaystyle\int_5^9 \left(\pi\left(3 + \sqrt{9-y}\right)^2 - \pi\left(3 - \sqrt{9-y}\right)^2\right) dy$

4. Let I be the exact value of $\int_0^2 (\sin(x) + \cos(2x))\, dx$ (you do not have to find I).

4A: Use an appropriate graph on your calculator to estimate the value of K_1 you should use in "theorem 3" to find the maximum possible error if a LHS is used as an approximation for I. (Give K_1 to **two** places after the decimal point).

$K_1 = \boxed{1.33}$ by looking at the graph of $f'(x) = (\sin x + \cos 2x)' = (\cos x - 2\sin 2x)$ on $[0, 2]$

4B: For the correct value of K_1 in (4A), find that maximum error guaranteed by theorem 3, if LHS(10) is used to approximate I. (Answer using **five** digits after the decimal point). Show the formula you use in your work.

guaranteed max. error is $\leq \dfrac{K_1(b-a)^2}{2n} = \dfrac{1.33(2-0)^2}{2 \cdot 10} = \dfrac{1.33 \times 4}{20}$

$= \dfrac{1.33 \times 2}{10}$

$= \dfrac{2.66}{10} = 0.266$

4C: Use an appropriate graph to find the value of K_2 you should use in theorem 3 to find an error bound for either a TRAP or MID approximation of I. **Fact:** $(\sin(x) + \cos(2x))'' = -\sin(x) - 4\cos(2x)$

$K_2 = \boxed{4.04}$ be careful: it looks like $K_2 = 4$ will do – and I thought so too – but closer inspection shows -4 is NOT the local min value!!

4D: Use the value of K_2 obtained in (4C) to find the smallest value of n you can choose so that theorem 3 guarantees MID(n) will be within 0.001 of I. *Show all your computations.*

We need $\dfrac{4.04(2-0)^3}{24 n^2} \leq 0.001$

or, $\dfrac{32.32}{24 \cdot 0.001} \leq n^2$

or, $1346.\overline{6} \leq n^2$

or $36.69\ldots \leq n$

since n must be an integer, we take $n = 37$

[note you will get $n = 37$ using $K_2 = 4$ as well]

4E: Find MID(n) for the value of n from (4D) to seven digits after the decimal point.

$\text{MID}(37) = 1.0377337$ (to 7 digits)

4F: Fact: $I = 1 - \cos(2) + (1/2)\sin(4)$. What is (to seven places after the decimal point) the actual error in using the answer to (4E)? (That is, find $|I - \text{MID}(n)|$ for the right value of n).

$I = 1 - \cos(2) + \frac{1}{2}\sin(4) = 1.0377456$ (to 7 digits)

So $|I - \text{MID}(n)| = |1.0377456 - 1.0377337|$

$= 0.0000119$ (note this is a much smaller error than 0.001, the max error possible according to thm 3.)

5. The following table of velocities $v(t)$ in feet per second at various times t for some moving object was recorded during an experiment:

t	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$v(t)$	-27	-21.1	-15.4	-10.3	-5.8	-2.0	1.3	4.0	6.4	8.4	10.1

only THIS info is used in 5B!

5A. What is the physical meaning of $\int_1^4 v(t)\,dt$?

the net, or total, change in POSITION of the object from $t=1$ to $t=4$

("DISTANCE" is WRONG)

5B. For the integral $\int_1^4 v(t)\,dt$ in (5A), estimate LHS(n), RHS(n), TRAP(n) and MID(n) for the maximum possible number of subintervals n in each case, using only the information available in the table. Clearly label all your answers!

We can find LHS(6) from the info in the box:

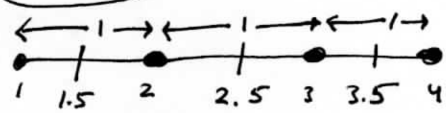
it's $(-15.4 * \Delta t) + (-10.3 * \Delta t) + (-5.8 * \Delta t) + (-2.0 * \Delta t) + (1.3 * \Delta t) + (4.0 * \Delta t)$ where $\Delta t = 0.5$, so

$LHS(6) = 0.5(-15.4 - 10.3 - 5.8 - 2 + 1.3 + 4) = -14.1$ (ft)

$RHS(6) = 0.5(-10.3 - 5.8 - 2 + 1.3 + 4 + 6.4) = -3.2$ "

$TRAP(6) = \dfrac{LHS(6) + RHS(6)}{2} = \dfrac{(-14.1) + (-3.2)}{2} = -8.65$ "

There's only enough info in the table to find MID(3)



using these 3 intervals, and midpts 1.5, 2.5 & 3.5 we get

$MID(3) = v(1.5)\cdot\Delta t + v(2.5)\cdot\Delta t + v(3.5)\cdot\Delta t$ where $\Delta t = 1$ $\therefore$

$\therefore\ MID(3) = (-10.3 \cdot 1) + (-2 \cdot 1) + (4.0 \cdot 1) = (-10.3 - 2.1 + 4) = -8.3$ ft

5C. The table suggests that $v(t)$ is an increasing, concave-down function on [0, 5]. Suppose it is, and there was enough information to find $I = \int_0^5 v(t)\,dt$ for each of LHS(n), RHS(n), TRAP(n) and MID(n) with $n = 25$. From smallest to largest, put these numbers in order: I, LHS(n), RHS(n), TRAP(n) and MID(n).

$$\boxed{LHS < TRAP < I < MID < RHS}$$

6A. Find $\int_1^4 \frac{\cos(8\sqrt{x})}{\sqrt{x}}\,dx$ by the method of substitution, using appropriate notation throughout.

let $u = 8\sqrt{x} = 8x^{1/2}$, then $du = 8\left(\frac{1}{2}x^{-1/2}\right)dx = 4x^{-1/2}dx = \frac{4}{\sqrt{x}}\,dx$

also $\begin{cases} x=1 \Rightarrow u=8 \\ x=4 \Rightarrow u=8\cdot 2=16 \end{cases}$

integral becomes: $\frac{1}{4}\int_1^4 \cos(8\sqrt{x})\frac{4}{\sqrt{x}}\,dx$

$$= \frac{1}{4}\int_8^{16} \cos u \; du = \frac{1}{4}\left(\sin u \Big|_8^{16}\right) = \boxed{\frac{1}{4}\left(\sin 16 - \sin 8\right)}$$

(which is $\approx \frac{1}{4}(-1.27726\ldots) \approx -0.319315\ldots$)

6B. In particular, what are the limits on the integral in (6A) after the appropriate substitution is made?

$\int_8^{16}$