Math 106 Winter 2013

Test 1 (50 points)

Name: Solutions.

Show all your work to receive full credit for a problem.

Do not use the calculator integral function. Whenever possible, find the exact values of integrals by finding antiderivatives or using the table of integrals.

When you use a formula from the table of integrals, mention the formula number and the value(s) of any constant(s) that you may need.

Round off your answers to four decimal places.

Include units in your answers wherever possible.

There are seven questions. Questions are printed on both sides of a page.

You may use any of the following facts:

Arclength$=\int_a^b \sqrt{1+(f'(x))^2}\,dx$

$|I-L_n| \leq \dfrac{K_1(b-a)^2}{2n}$ $\qquad$ $|I-R_n| \leq \dfrac{K_1(b-a)^2}{2n}$

$|I-T_n| \leq \dfrac{K_2(b-a)^3}{12n^2}$ $\qquad$ $|I-M_n| \leq \dfrac{K_2(b-a)^3}{24n^2}$

Below are product rule, quotient rule and chain rule for derivatives.

$(uv)' = u'v+uv'$ $\qquad$ $\left(\dfrac{u}{v}\right)' = \dfrac{vu'-uv'}{v^2}$ $\qquad$ $(f\circ g)'(x) = f'(g(x))\,g'(x)$

1. (8 points) Evaluate the following integral exactly. (You may use formulas 1-18 only from the table of integrals for this problem.)

$$\int_0^{\pi/6} \frac{\cos(3x)}{4+\sin^2(3x)}\,dx$$

$u = \sin(3x)$

$du = 3\cos(3x)\,dx$.

New limits: When $x=0$, $u=0$

When $x=\frac{\pi}{6}$, $u=1$.

$$\int_0^{\pi/6} \frac{\cos(3x)}{4+\sin^2(3x)}\,dx = \int_0^1 \frac{1}{4+u^2}\cdot\frac{1}{3}\,du$$

$$= \frac{1}{3}\int_0^1 \frac{1}{4+u^2}\,du$$

$$= \frac{1}{3}\cdot\frac{1}{2}\arctan\left(\frac{u}{2}\right)\Big]_0^1$$ Formula 13 from table with $a=2$.

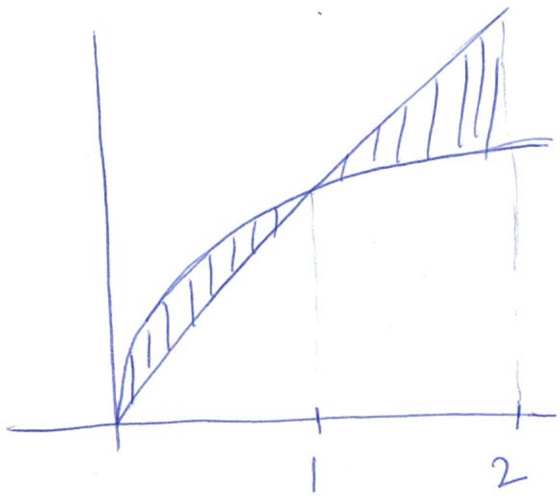
$$= \frac{1}{6}\left[\arctan\left(\frac{1}{2}\right) - \arctan(0)\right]$$

$$= \frac{1}{6}\arctan(0.5).$$

2. (6 points) Write (but do not evaluate) an integral to find the area of the region bounded by the curves $y = x$, $y = \sqrt{x}$, $x = 0$ and $x = 2$.

$x = \sqrt{x}$. $x^2 = x$. $x^2 - x = 0$. $x(x-1) = 0$

$x = 0, 1$.



Shaded area

$$= \int_0^1 (\sqrt{x} - x)\,dx + \int_1^2 (x - \sqrt{x})\,dx.$$

3. (8 points) An ant is walking along the curve $y = 3x^2$. Find the exact distance traveled by the ant as it walks from the point $(0, 0)$ to the point $(1, 3)$ along the curve.

$f(x) = 3x^2$. $f'(x) = 6x$.

Exact distance $= \int_0^1 \sqrt{(6x)^2 + 1}\,dx$.

To use formula 36 ~~for~~ from table, substitute $u = 6x$, $du = 6\,dx$.

New limits: When $x = 0, u = 0$. When $x = 1, u = 6$.

$$\int_0^1 \sqrt{(6x)^2 + 1}\,dx = \int_0^6 \sqrt{u^2+1} \cdot \frac{1}{6}\,du.$$

$$= \frac{1}{6}\left[\frac{1}{2}\left(u\sqrt{u^2+1} + \ln\left|u + \sqrt{u^2+1}\right|\right)\right]_0^6$$ Formula 36 with $p = 1$.

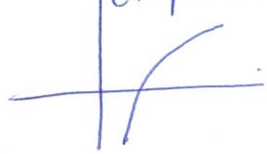
$$= \frac{1}{12}\left[6\sqrt{37} + \ln(6 + \sqrt{37})\right].$$

4. (6 points) Let $I = \int_{0.5}^{2} x \ln x \, dx$. What is the least value of n which guarantees that R_n approximates I within ± 0.01? Justify your answer.

$$|I - R_n| \le \frac{K_1 (b-a)^2}{2n}. \qquad a = 0.5,\ b = 2.$$

Find K_1: $f'(x) = \ln x + 1$.

Graph of $f'(x)$.



So $K_1 = f'(2) = 1 + \ln 2$.

$$|I - R_n| \le \frac{(1+\ln 2)(2-0.5)^2}{2n} \le 0.01.$$

So $2n \geq \frac{(1+\ln 2)(1.5)^2}{0.01}$ $\quad n \geq \frac{(1+\ln 2)(1.5)^2}{0.02}$ ie $n \geq 190.4791$.

So least value of n is 191.

5. (6 points) Let $I = \int_1^3 e^{-f(x)} \, dx$ where $f(x)$ is a function with the following properties: $6 \le f'(x) \le 8$ and $-4 \le f''(x) \le 0$ for all x in the interval $[1, 3]$.

(a) Does L_{50} overestimate or underestimate I? Justify your answer.

Let $g(x) = e^{-f(x)}$ Need to decide if $g(x)$ is increasing or decreasing on $[1,3]$.

$$g'(x) = \underbrace{e^{-f(x)}}_{\text{always positive}} \cdot \underbrace{(-f'(x))}_{\text{negative on } [1,3] \text{ since } f'(x) > 0}.$$

So $g'(x) < 0$ on $[1,3]$. Hence g is decreasing on $[1,3]$. So L_{50} overestimates I.

(b) Does M_{50} overestimate or underestimate I? Justify your answer.

Need to decide if $g(x)$ is concave up or down on $[1,3]$.

$$g''(x) = e^{-f(x)}(-f''(x)) + e^{-f(x)}(-f'(x))(-f'(x))$$

$$\text{ie } g''(x) = \underbrace{e^{-f(x)}}_{\text{positive}} \underbrace{(-f''(x))}_{\geq 0} + \underbrace{e^{-f(x)} (f'(x))^2}_{\text{positive}}.$$

So $g''(x) > 0$ on $[1,3]$. So g is concave up on $[1,3]$.

Hence M_{50} underestimates I.

6. (8 points) Solve the IVP: $y' = \dfrac{1}{y^2 - 4y^2x}$, $y(0) = 2$. (You may use formulas 1-18 only from the table of integrals for this problem.)

$$\frac{dy}{dx} = \frac{1}{y^2(1-4x)}$$

$$y^2\,dy = \frac{1}{1-4x}\,dx.$$

$$\int y^2\,dy = \int \frac{1}{1-4x}\,dx.$$

$$\frac{y^3}{3} = \int \frac{1}{1-4x}\,dx.$$

To find $\int \frac{1}{1-4x}\,dx$:

$u = 1-4x.$ $\quad du = -4\,dx.$

$$\int \frac{1}{1-4x}\,dx = \int \frac{1}{u}\frac{1}{(-4)}\,du = \frac{1}{-4}\ln|u| + C = -\frac{1}{4}\ln|1-4x| + C.$$

Hence $\dfrac{y^3}{3} = -\dfrac{1}{4}\ln|1-4x| + C$.

So $y^3 = -\dfrac{3}{4}\ln|1-4x| + 3C$.

$$y = \sqrt[3]{-\frac{3}{4}\ln|1-4x| + 3C}.$$

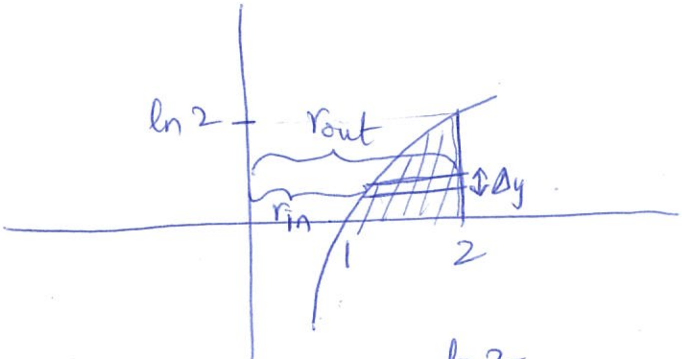
Use initial condition to find C:

$x=0,\ y=2$. So $2 = \sqrt[3]{-\frac{3}{4}(0) + 3C}$. ie $8 = 3C$ $\quad C = \frac{8}{3}$.

So $y = \sqrt[3]{8 - \frac{3}{4}\ln|1-4x|}$.

7. (8 points)

(a) Sketch the region bounded by the curve $y = \ln x$, and the lines $x = 2$ and $y = 0$. Write (but do not evaluate) an integral to find the volume of the solid that is formed when the region is rotated about the $y-$axis.



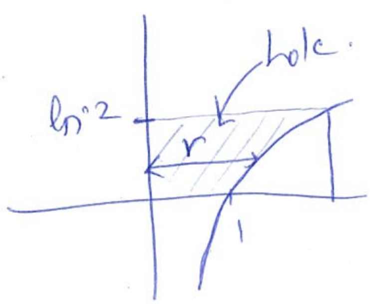
$r_{out} = 2$.

$r_{in} = x = e^y$.

$$\text{Volume} = \int_0^{\ln 2} (\pi r_{out}^2 - \pi r_{in}^2)\, dy$$

$$= \int_0^{\ln 2} (4\pi - \pi e^{2y})\, dy.$$

(b) Write (but do not evaluate) an integral to find the volume of the hole in the solid in part(a).



$r = x = e^y$

$$\int_0^{\ln 2} \pi e^{2y}\, dy.$$