MATH106A,B CALCULUS II - PROF. P. WONG

EXAM I - JANUARY 31, 2014

NAME:

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

Advice: DON'T spend too much time on a single problem.

Problems	**Maximum Score**	**Your Score**
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
Total	100	

1.(10 pts.)(a) Evaluate the indefinite integral (be sure to show all your work)

$$\int x^2(x^3+6)^8\ dx.$$

Let $u = x^3+6$ **so that** $du = 3x^2\ dx$ **or** $x^2\ dx = \frac{1}{3}du$**. Now,**

$$\begin{aligned}\int x^2(x^3+6)^8\ dx &= \int u^8 \cdot \frac{1}{3}\ du = \frac{1}{3}\int u^8\ du \\ &= \frac{1}{3}\cdot\frac{u^9}{9} + C \\ &= \frac{(x^3+6)^9}{27} + C.\end{aligned}$$

(10 pts.) (b) Find the **exact value** of the definite integral (be sure to show all your work)

$$\int_0^4 \sqrt{x^2+x}(2x+1)\ dx.$$

Let $u = x^2+x$ **so that** $du = (2x+1)\ dx$**. When** $x=0, u=0$ **and when** $x=4, u=20$**. It follows that**

$$\begin{aligned}\int_0^4 \sqrt{x^2+x}(2x+1)\ dx &= \int_0^{20}\sqrt{u}\ du = \left.\frac{u^{3/2}}{3/2}\right|_0^{20} \\ &= \frac{2}{3}(20)^{3/2}.\end{aligned}$$

2. Consider the region A bounded by the curve $y = x^3 - x^2 - 2x + 5$ and the line $y = 5$.

(15 pts.) Find the **exact area** of the region A.

The curve and the line intersect at points where $x^3 - x^2 - 2x + 5 = 5$ **or** $x^3 - x^2 - 2x = 0$**. It follows that the intersections occur when** $x(x-2)(x+1) = 0$ **or when** $x = 0, 2, -1$**.**

Between $x = -1$ **and** $x = 0$ **the curve** $y = x^3 - x^2 - 2x + 5$ **lies above the line** $y = 5$ **whereas the line** $y = 5$ **lies above the curve** $y = x^3 - x^2 - 2x + 5$ **between** $x = 0$ **and** $x = 2$**. Thus the area of the region** A **bounded by the curve and the line is given by**

$$\int_{-1}^{0} (x^3 - x^2 - 2x + 5) - 5 \ dx + \int_{0}^{2} 5 - (x^3 - x^2 - 2x + 5) \ dx$$
$$= \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right) \Big|_{-1}^{0} + \left(-\frac{x^4}{4} + \frac{x^3}{3} + x^2 \right) \Big|_{0}^{2}$$
$$= -\left(\frac{1}{4} + \frac{1}{3} - 1 \right) + \left(-4 + \frac{8}{3} + 4 \right) = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}.$$

(5 pts.) Let B be the region bounded by the same two curves and the line $x = 2$ that lies in the first quadrant. Write a definite integral (do not evaluate) representing the area of region B. [By area, we mean the usual *physical* area not *signed* area.]

The region B **is the same as that of region** A **except that it lies in the first quadrant, i.e.,** x **lies between** 0 **and** 2**. It follows that the area of region** B **is given by**

$$\int_{0}^{2} 5 - (x^3 - x^2 - 2x + 5) \ dx = \int_{0}^{2} -x^3 + x^2 + 2x \ dx.$$

3. (12 pts.) Consider a function g on the interval $[0, 2]$.

x	0	0.5	1	1.5	2	2.5	3
$g(x)$	-1	1	3	2	-1	-3	-2

Find R_6, M_3 using the right-hand sum and the mid-point rule respectively for estimating the definite integral $\int_0^3 g(x)\ dx$.

For the right hand sum with $n = 6$, the length of each subinterval is $\Delta x = 0.5$ so that

$$R_6 = [1 + 3 + 2 - 1 - 3 - 2] \cdot (0.5) = 0.$$

For the mid-point rule with $n = 3$, the length of each subinterval is $\Delta x = 1$ so that

$$M_3 = [1 + 2 - 3] \cdot (1) = 0.$$

(8 pts.)(b) Recall that the error committed by using the left hand sum approximation L_n is less than or equal to $\frac{K_1 \cdot (b-a)^2}{2n}$ where $|f'(x)| \leq K_1$ for some constant K_1 over the interval $[a, b]$. Use this result to give an upper bound for the error committed by L_{10} for

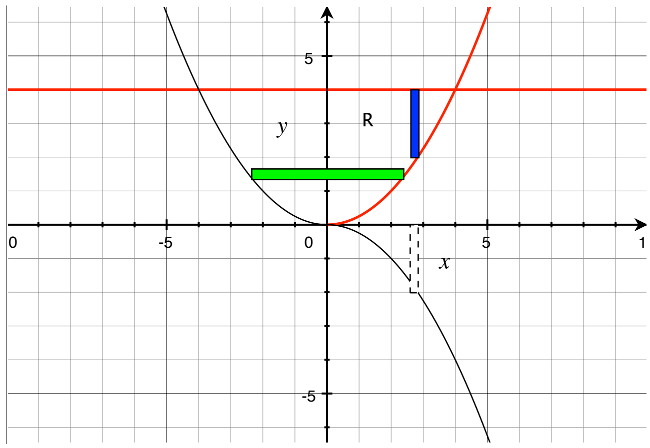
$$I = \int_4^6 e^{-x} \sin x\ dx.$$

Recall that $K_1 \geq |f'(x)|$ for all $x, 4 \leq x \leq 6$. Since $f(x) = e^{-x} \sin x$, $f'(x) = -e^{-x} \sin x + e^{-x} \cos x = e^{-x}(\cos x - \sin x)$. Over the interval $[4, 6]$, the maximum of $|f'(x)|$ is approximately 0.00898 (this maximum does not occur at the endpoints). We now may choose $K_1 = 0.009$. It follows that the error committed by L_{10} is less than or equal to

$$\frac{K_1 \cdot (b-a)^2}{2n} = \frac{(0.009) \cdot (6-4)^2}{20} = 0.0018.$$

4. Let R be the region bounded by the curve $x = 2\sqrt{y}$, the line $y = 4$, and the line $x = 0$.

(12 pts.) (a) Find the **exact volume** of the solid obtained from rotating the region R around the y-axis. [Hint: sketch a picture of the region R first.]



By using horizontall slices (green), the volume of a typical slice is approximately $\pi(2\sqrt{y})^2 \cdot \Delta y$. Since the curve $x = 2\sqrt{y}$ intersects the line $y = 4$ at the point $(4, 4)$. It follows that the volume of the solid is given by

$$\begin{aligned} V &= \int_0^4 \pi(2\sqrt{y})^2 \, dy \\ &= \pi \int_0^4 4y \, dy \\ &= 4\pi \frac{y^2}{2}\Big|_0^4 = 32\pi. \end{aligned}$$

(8 pts.) (b) Set up (do not evaluate) a definite integral representing the volume of the solid obtained from rotating the region R around the x-axis.

Using vertical slices (blue), the volume of a typical slice is approximately $[\pi(4)^2 - \pi(\frac{x^2}{4})^2] \cdot \Delta x$. It follows that the volume of the resulting solid is given by

$$\int_0^4 [\pi(4)^2 - \pi(\frac{x^2}{4})^2] \, dx = \pi \int_0^4 16 - \frac{x^4}{16} \, dx.$$

5. Consider the initial value problem

$$\frac{dy}{dx} = \frac{x}{y}$$

with $y(1) = 1$.

(10 pts.)(a) Use the technique of separation of variables to solve the Initial Value Problem.

By separating the variables, we obtain

$$y\ dy = x\ dx \qquad \textbf{which implies} \qquad \int y\ dy = \int x\ dx.$$

It follows that

$$\frac{y^2}{2} = \frac{x^2}{2} + C.$$

Since $y(1) = 1$, we have $\frac{1}{2} = \frac{1}{2} + C$ which implies that $C = 0$. Now, solving for y yields $y = \pm x$. However the initial condition $y(1) = 1$ forces $y = x$.

(10 pts.)(b) Find the arc length of the portion of the graph of $f(x) = 4x^{3/2}$ between $x = 0$ and $x = 1$.

Note that $f'(x) = 4 \cdot \frac{3}{2}x^{1/2} = 6\sqrt{x}$ so $[f'(x)]^2 = 36x$. Now the arclength is given by

$$\begin{aligned}\int_0^1 \sqrt{1 + [f'(x)]^2}\ dx &= \int_0^1 \sqrt{1 + 36x}\ dx \\ &= \int_1^{37} \frac{\sqrt{u}}{36}\ du \qquad \textbf{where } u = 1 + 36x \\ &= \frac{1}{36} \cdot \frac{u^{3/2}}{3/2}\Big|_1^{37} \\ &= \frac{(37)^{3/2} - 1}{54}.\end{aligned}$$