Math 106 Winter 2014

Test 1 (50 points)

Name: Solutions

Show all your work to receive full credit for a problem. Points will be taken off if you do not show how you arrived at your answer, even if the final answer is correct.

Please keep your written answers brief; be clear and to the point. Points will be taken off for rambling and for incorrect or irrelevant statements and for multiple solutions to the same problem.

Do not use the calculator integral function. Whenever possible, find the exact values of integrals by finding antiderivatives or using the table of integrals.

When you use a formula from the table of integrals, mention the formula number and the value(s) of any constant(s) that you may need.

Give exact answers. If needed, round off your answers to four decimal places.

Include units in your answers wherever possible.

There are six questions. Questions are printed on both sides of a page.

You may use any of the following facts:

Arclength$=\displaystyle\int_a^b \sqrt{1+(f'(x))^2}\,dx$

$$|I-L_n| \le \frac{K_1(b-a)^2}{2n} \qquad |I-R_n| \le \frac{K_1(b-a)^2}{2n}$$

$$|I-T_n| \le \frac{K_2(b-a)^3}{12n^2} \qquad |I-M_n| \le \frac{K_2(b-a)^3}{24n^2}$$

Below are product rule, quotient rule and chain rule for derivatives.

$$(uv)' = u'v+uv' \qquad \left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2} \qquad (f\circ g)'(x) = f'(g(x))\,g'(x)$$

1. (8 points) Evaluate the following integral exactly. (You may use formulas 1-18 only from the table of integrals for this problem.)

$$\int_4^9 \frac{7e^{\sqrt{x}}-1}{\sqrt{x}}\,dx\,.$$

$u=\sqrt{x}$. $\quad du = \frac{1}{2}x^{-1/2}dx = \frac{1}{2\sqrt{x}}dx$.

So $dx = 2\sqrt{x}\cdot du$

$$\int \frac{7e^{\sqrt{x}}-1}{\sqrt{x}}dx = \int \frac{7e^{u}-1}{\cancel{\sqrt{x}}}\cdot 2\cancel{\sqrt{x}}\,du = 2\int (7e^{u}-1)\,du$$

$$= 2[7e^{u}-u]+C$$

$$= 2(7e^{\sqrt{x}}-\sqrt{x})+C$$

$$\int_4^9 \frac{7e^{\sqrt{x}}-1}{\sqrt{x}}dx = 2(7e^{\sqrt{x}}-\sqrt{x})\Big]_4^9$$

$$= 2(7e^{3}-3-7e^{2}+2)$$

$$= 2(7e^{3}-7e^{2}-1)\,.$$

$$= \boxed{14e^{3}-14e^{2}-2}\,.$$

OR Let $u=e^{\sqrt{x}}$. $\quad du = e^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}dx$. $\quad dx = \frac{2\sqrt{x}}{e^{\sqrt{x}}}du = \frac{2\sqrt{x}}{u}du$

$$\int \frac{7e^{\sqrt{x}}-1}{\sqrt{x}}dx = \int \frac{7u-1}{\cancel{\sqrt{x}}}\cdot\frac{2\cancel{\sqrt{x}}}{u}du = 2\int \frac{7u-1}{u}du = 2\int\left(7-\frac{1}{u}\right)du$$

$$= 2(7u-\ln|u|) = 2(7e^{\sqrt{x}}-\sqrt{x})+C$$

Now proceed as before.

$$\text{OR}\quad \int_4^9 \frac{7e^{\sqrt{x}}-1}{\sqrt{x}}dx = \underbrace{\int_4^9 \frac{7e^{\sqrt{x}}}{\sqrt{x}}dx}_{\text{For this integral, use any one of the above substitutions}} - \underbrace{\int_4^9 \frac{1}{\sqrt{x}}dx}_{\text{This integral can be evaluated directly}}\,.$$

2. (8 points) Solve the DE: $y' \sin^2(2x) = y^2 \cos(2x)$. (You may use formulas 1-18 only from the table of integrals for this problem.)

$$\frac{dy}{dx} \cdot \sin^2(2x) = y^2 \cos(2x).$$

$$\frac{dy}{y^2} = \frac{\cos(2x)}{\sin^2(2x)}\, dx.$$

$$\int y^{-2}\, dy = \int \frac{\cos(2x)}{\sin^2(2x)}\, dx.$$

$$\int y^{-2}\, dy = \frac{y^{-1}}{-1} = \frac{-1}{y}$$

$$\int \frac{\cos(2x)}{\sin^2(2x)}\, dx \qquad u = \sin(2x)$$
$$du = 2\cos(2x)\, dx. \qquad dx = \frac{1}{2\cos(2x)}\, du.$$

$$= \int \frac{\cancel{\cos(2x)}}{u^2} \frac{1}{2\cancel{\cos(2x)}}\, du$$

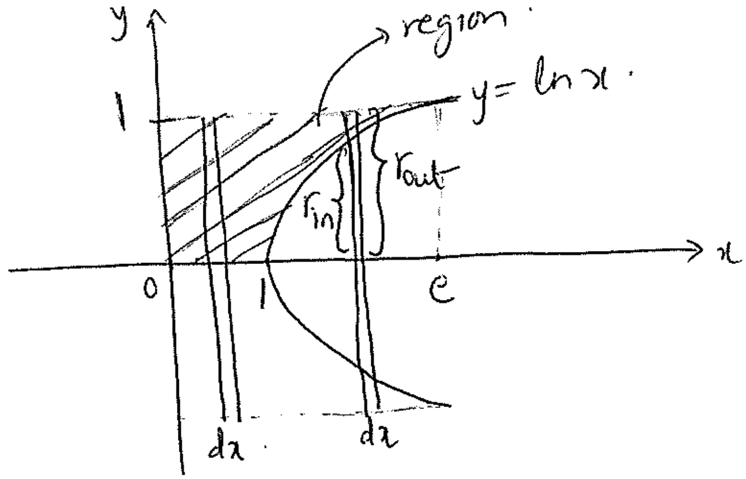
$$= \frac{1}{2} \int u^{-2}\, du = \frac{1}{2} \frac{u^{-1}}{-1} = \frac{-1}{2u} = \frac{-1}{2\sin(2x)}$$

So $$-\frac{1}{y} = \frac{-1}{2\sin(2x)} + C.$$

$$\frac{1}{y} = \frac{1}{2\sin(2x)} - C.$$

$$y = \frac{1}{\frac{1}{2\sin(2x)} - C} = \frac{2\sin(2x)}{1 - 2C\sin(2x)}.$$

3. (7 points) Sketch the region bounded by the curve $y = \ln x$, the line $y = 1$, the x−axis and the y−axis. Write (but do not evaluate) an integral to find the volume of the solid obtained by rotating the region about the x−axis.



For $\underline{0 \le x < 1}$, slice is a disk with radius $= 1$.

Cross-section area $= \pi(1)^2 = \pi$.

Volume of slice $= \pi\, dx$.

Volume of solid between 0 and 1 is $\int_0^1 \pi\, dx$.

For $\underline{1 \le x \le e}$, slice is a washer.

$r_{in} = y = \ln x$

$r_{out} = 1$.

Cross-section area $= \pi r_{out}^2 - \pi r_{in}^2 = \pi(1)^2 - \pi(\ln x)^2$

$= \pi\left(1 - (\ln x)^2\right)$.

Volume of slice $= \pi\left(1 - (\ln x)^2\right) dx$.

Volume of solid between 1 and e is $\int_1^e \pi\left(1 - (\ln x)^2\right) dx$.

Volume of entire solid is

$$\int_0^1 \pi\, dx + \int_1^e \pi\left(1 - (\ln x)^2\right) dx$$

4. (8 points) Let $I = \int_2^6 f(x)\,dx$ where $f(x)$ is a function with the following properties:

$0 \le f'(x) \le 10$ and $-3 \le f''(x) \le 1$ for all x in the interval $[2, 6]$.

What is the least value of n which guarantees that T_n approximates I within ± 0.05? Justify your answer.

We want n such that $|I - T_n| \le 0.05$.

From theorem, $|I - T_n| \le \dfrac{K_2 (b-a)^3}{12n^2}$.

So it is enough to find n such that $\dfrac{K_2 (b-a)^3}{12n^2} \le 0.05$.

$a = 2$, $b = 6$.

In $[2,6]$, $-3 \le f''(x) \le 1$. So $|f''(x)| \le 3$. $K_2 = 3$.

$\dfrac{3(6-2)^3}{12n^2} \le 0.05$. $\dfrac{16}{n^2} \le 0.05$ So $n^2 \ge \dfrac{16}{0.05} = 320$.

So $n \ge 17.8885$. Hence least value of n is $\underline{18}$.

5. (7 points) Find the exact length of the curve $y = 4 - 5x^2$ from $x = 0$ to $x = 1$.

$f(x) = 4 - 5x^2$. $f'(x) = -10x$.

~~$f'(x) = 4-$~~

Exact length $= \int_0^1 \sqrt{1 + (-10x)^2}\,dx = \int_0^1 \sqrt{1 + 100x^2}\,dx$.

$u = 10x$. $du = 10\,dx$. When $x = 0$, $u = 0$. When $x = 1$, $u = 10$.

$\int_0^{10} \sqrt{1 + u^2} \cdot \dfrac{du}{10} = \dfrac{1}{10} \int_0^{10} \sqrt{u^2 + 1}\,du$.

Use formula 36 from table with $p = 1$.

So length $= \dfrac{1}{10}\left[\dfrac{1}{2}\left(u\sqrt{u^2+1} + \ln|u + \sqrt{u^2+1}|\right)\right]_0^{10}$.

$= \dfrac{1}{20}\left(10\sqrt{101} + \ln|10 + \sqrt{101}| - 0\right) = \dfrac{1}{20}\left(10\sqrt{101} + \ln|10 + \sqrt{101}|\right)$

6. (12 points) Water flows into a storage tank at a rate of $r(t)$ ft^3/min, where t is the number of minutes since the water starts to flow in. The table below gives data for t and $r(t)$. Assume that $r(t)$ is increasing on the interval $[0, 7]$. Let $I = \int_1^7 r(t)\,dt$.

t	1	2	3	4	5	6	7
$r(t)$	30	33	34	36	37	39	42

(a) What does the integral $\int_1^7 r(t)\,dt$ represent?

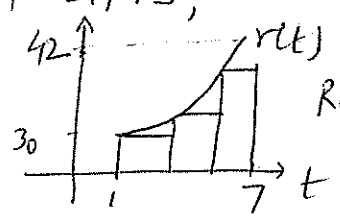
The integral represents the amount of water (in cubic feet) that flows into the tank between 1 and 7 minutes.

(b) Use the table to find the approximation L_3 to I. Does L_3 underestimate or overestimate I? Justify your answer.

Width of each subinterval $= \frac{7-1}{3} = 2$.

$L_3 = (r(1)+r(3)+r(5)) \cdot 2 = (30+34+37) \cdot 2 = 202$ ft^3.

Since $r(t)$ is increasing on $[1,7]$, L_3 underestimates I.



Rectangles for L_3.

(c) Indicate whether the following statement must be true, cannot be true or may be true. Justify your answer.

$$M_{20} < I.$$

This statement may be true. Since we ~~do not~~ cannot say anything with certainty about the concavity of $r(t)$, we do not know whether M_{20} underestimates or overestimates I