

### Math 106: Review for Exam I - SOLUTIONS

1. **Find the following.** [Substitution tip: usually let  $u =$  a function that's "inside" another function, especially if  $du$  (possibly off by a multiplying constant) is also present in the integrand.]

(a) Let  $u = \sqrt{x}$ , so  $du = \frac{dx}{2\sqrt{x}}$  and  $2 du = \frac{dx}{\sqrt{x}}$ .

Now we'll change the limits.

If  $x = 1$ , then  $u = \sqrt{1} = 1$  and if  $x = 4$ , then  $u = \sqrt{4} = 2$ .

$$\begin{aligned}\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx &= \int_1^2 e^u \cdot 2 du \\ &= 2e^u \Big|_1^2 \\ &= 2e^2 - 2e (\approx 9.342)\end{aligned}$$

(b) Let  $u = \cos(5x)$ , so  $du = -5 \sin(5x) dx$  and  $-\frac{du}{5} = \sin(5x) dx$ .

Now we'll change the limits.

If  $x = \pi$ , then  $u = \cos(5 \cdot \pi) = -1$  and if  $x = 2$ , then  $u = \cos(5 \cdot 2\pi) = 1$ .

$$\begin{aligned}\int_{\pi}^{2\pi} \cos^7(5x) \sin(5x) dx &= \int_{-1}^1 u^7 \cdot \frac{-du}{5} \\ &= -\frac{1}{5} \int_{-1}^1 u^7 du \\ &= -\frac{1}{5} \frac{u^8}{8} \Big|_{-1}^1 \\ &= -\frac{1}{40} (1^8 - (-1)^8) \\ &= 0\end{aligned}$$

(c) Use  $u = x^3$ , so  $du = 3x^2 dx$  and  $\frac{du}{3} = x^2 dx$ .

$$\begin{aligned}\int \frac{7x^2}{1+x^6} dx &= 7 \int \frac{\frac{du}{3}}{1+u^2} \\ &= \frac{7}{3} \arctan u + C \\ &= \frac{7}{3} \arctan(x^3) + C\end{aligned}$$

(d) Use  $u = 10 - x$ , so  $du = -dx$  and  $dx = -du$ .

$$\begin{aligned} \int x\sqrt{10-x} dx &= \int (10-u)\sqrt{u}(-du) && \text{Since } u = 10 - x, \text{ we know } x = 10 - u. \\ &= \int (u-10)\sqrt{u} du \\ &= \int (u^{3/2} - 10u^{1/2}) du \\ &= \frac{2}{5}u^{5/2} - \frac{20}{3}u^{3/2} + C \\ &= \frac{2}{5}(10-x)^{5/2} - \frac{20}{3}(10-x)^{3/2} + C \end{aligned}$$

2. If  $f(x)$  is decreasing and concave up, put the following quantities in ascending order.

$$L_{100}, R_{100}, T_{100}, M_{100}, \int_a^b f(x) dx \qquad R_{100} < M_{100} < \int_a^b f(x) dx < T_{100} < L_{100}$$

What can you say with certainty about where  $S_{200}$  would fit into your list above?

It would be somewhere between  $M_{100}$  and  $T_{100}$  but we don't know how it compares to  $\int_a^b f(x) dx$ .

3. Find the best possible left, right, midpoint, trapezoidal, and Simpson's approximations to  $\int_4^{12} f(t) dt$  given the data in the table below.

$t$	4	6	8	10	12
$f(t)$	15	11	8	4	3

$$L_4 = (15 + 11 + 8 + 4)(2) = 76 \qquad R_4 = (11 + 8 + 4 + 3)(2) = 52 \qquad T_4 = \frac{L_4 + R_4}{2} = 64$$

We cannot compute  $M_4$  because it requires the values of  $f$  at  $x = 5, 7, 9$ , and  $11$ . Instead, we do  $M_2$ .

$$M_2 = (11 + 4)(4) = 60$$

$$\text{Now, to find } S_4, \text{ we need } T_2 = \frac{L_2 + R_2}{2} = \frac{(15 + 8)(4) + (8 + 3)(4)}{2} = 68.$$

$$S_4 = \frac{2M_2 + T_2}{3} = \frac{2(60) + 68}{3} = \frac{188}{3} = 62.\bar{6}$$

4. Find bounds for each of the following errors if  $I = \int_2^7 \ln x dx$ .

$$(a) |I - L_{100}| \leq \frac{K_1(b-a)^2}{2n} = \frac{\frac{1}{2}(7-2)^2}{2(100)} = \frac{1}{16}$$

$$K_1 = \max \text{ of } |f'(x)| \text{ on } [2, 7] = \max \text{ of } \frac{1}{x} \text{ on } [2, 7] = \frac{1}{2} \text{ (occurs at } x = 2)$$

$$(b) |I - T_{100}| \leq \frac{K_2(b-a)^3}{12n^2} = \frac{\frac{1}{4}(7-2)^3}{12(100)^2} = \frac{1}{3840}$$

$$K_2 = \max \text{ of } |f''(x)| \text{ on } [2, 7] = \max \text{ of } \frac{1}{x^2} \text{ on } [2, 7] = \frac{1}{4} \text{ (occurs at } x = 2)$$

$$(c) |I - M_{100}| \leq \frac{K_2(b-a)^3}{24n^2} = \frac{\frac{1}{4}(7-2)^3}{24(100)^2} = \frac{1}{7680}$$

$K_2 =$  same as in previous part

5. If  $I = \int_2^7 \ln x \, dx$ , how many subdivisions are required to obtain a trapezoidal sum approximation with error of at most  $1/1,000,000$ ?

From part (b) above, we know that  $|I - T_n| \leq \frac{K_2(b-a)^3}{12n^2} = \frac{\frac{1}{4}(7-2)^3}{12n^2} = \frac{125}{48n^2}$ .

Thus, we want  $\frac{125}{48n^2} \leq \frac{1}{1,000,000}$ .

Multiplying each side by  $1,000,000n^2$  gives  $\frac{125,000,000}{48} \leq n^2$ .

Taking the square root of each side results in  $\sqrt{\frac{125,000,000}{48}} \leq n$ .

Since  $\sqrt{\frac{125,000,000}{48}} = 1613.743\dots$ , we must at least 1614 subdivisions.

6. Solve the differential equation  $dy/dx = 2xy + 6x$  if the solution passes through  $(0, 5)$ .

$$\frac{dy}{dx} = 2xy + 6x$$

$$\frac{dy}{dx} = 2x(y + 3)$$

$$\frac{dy}{y + 3} = 2x \, dx$$

Separate the variables.

$$\int \frac{dy}{y + 3} = \int 2x \, dx$$

$$\ln|y + 3| = x^2 + C$$

$$|y + 3| = e^{x^2 + C}$$

Exponentiate each side to remove the ln.

$$y + 3 = \pm e^C e^{x^2}$$

$|w| = z$  means  $w = \pm z$ .

$$y = -3 + Ae^{x^2}$$

Replace  $\pm e^C$  with  $A$ .

Now we use the initial condition  $y(0) = 5$  to find the value of  $A$ .

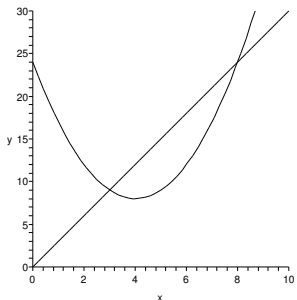
We have  $5 = -3 + Ae^0 \Rightarrow A = 8$ , so the solution is  $y = -3 + 8e^{x^2}$ .

7. Write integrals equal to

- (a) the arc length of  $y = x^2$  on the interval  $[1, 5]$

$$\text{arc length of } y = f(x) \text{ on } [a, b] = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx = \int_1^5 \sqrt{1 + (2x)^2} \, dx \ (\approx 24.395)$$

- (b) the area bounded by  $y = x^2 - 8x + 24$  and  $y = 3x$



First, find where the curves intersect.

$$\begin{aligned} x^2 - 8x + 24 &= 3x \\ x^2 - 11x + 24 &= 0 \\ (x - 3)(x - 8) &= 0 \\ &\Rightarrow x = 3, x = 8 \end{aligned}$$

Between  $x = 3$  and  $x = 8$ ,  $y = 3x$  is above  $y = x^2 - 8x + 24$ . (Plug in  $x = 5$  or graph to check.)

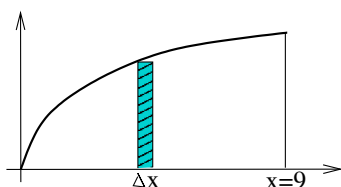
So, the area between them is

$$\int_3^8 [3x - (x^2 - 8x + 24)] dx.$$

[This equals  $125/6$ .]

8. Consider the region bounded by  $y = \sqrt{x}$ ,  $y = 0$ , and  $x = 9$ . Write an integral equal to the volume generated if this region is revolved about

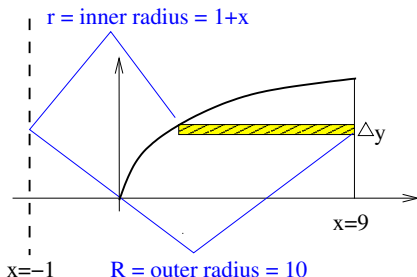
(a) the  $x$ -axis



$$\begin{aligned} \text{volume of slice} &\approx \pi r^2 \Delta x \\ &= \pi y^2 \Delta x \\ &= \pi (\sqrt{x})^2 \Delta x \\ &= \pi x \Delta x \end{aligned}$$

$$\text{total volume} = \pi \int_0^9 x dx$$

(b) the line  $x = -1$

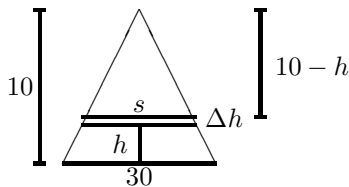
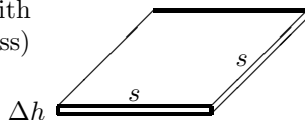


$$\begin{aligned} \text{volume of slice} &\approx \pi R^2 \Delta y - \pi r^2 \Delta y \\ &= \pi (10^2) \Delta y - \pi (1 + x)^2 \Delta y \\ &= \pi [100 - (1 + y^2)^2] \Delta y \end{aligned}$$

$$\text{total volume} = \pi \int_0^3 [100 - (1 + y^2)^2] dy$$

9. A pyramid has a square base 30 feet to a side and a height of 10 feet. Write an integral equal to the volume of the pyramid.

We slice horizontally, so each slice is a “box” with a square top and bottom and a height (thickness) of  $\Delta h$ ,



Similar triangles:  $\frac{10}{30} = \frac{10 - h}{s} \Rightarrow s = 3(10 - h)$ .

volume of slice  $\approx s^2 \Delta h \approx [3(10 - h)]^2 \Delta h$

$$\text{total volume} = \int_0^{10} [3(10 - h)]^2 dh$$