

1. Consider the following vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 6 \\ 4 \\ -1 \end{bmatrix} \text{ and } \mathbf{b}_1 = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

1a. Does the vector equation  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{b}_1$  have any solutions  $(x_1, x_2, x_3, x_4)$ ? If so, write the solutions in our standard notation with all "basic" variables in terms of the free variables (if there are any free variables). But if there are no solutions, explain why not. Show any augmented matrices and their RREF's that you use in answering this question.

The augmented matrix which represents this system is

$$\left[ \begin{array}{cccc|c} 3 & 2 & 5 & 6 & 7 \\ 4 & 4 & 4 & 4 & 4 \\ 5 & 7 & 1 & -1 & -3 \end{array} \right] \text{ which has RREF } \left[ \begin{array}{cccc|c} 1 & 0 & 3 & 4 & 5 \\ 0 & 1 & -2 & -3 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{which represents the system of equations } \begin{cases} x_1 + 3x_3 + 4x_4 = 5 \\ x_2 - 2x_3 - 3x_4 = -4 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 = 0 \end{cases}$$

$$\text{which we know has solutions } \begin{cases} x_1 = 5 - 3x_3 - 4x_4 \\ x_2 = -4 + 2x_3 + 3x_4 \\ \text{where } x_3 \text{ and } x_4 \\ \text{are free.} \end{cases}$$

[note this last eqn is solved by ANY  $x_1, x_2, x_3, x_4$  and is therefore useless]

1b. If there are solutions in (1a), set all the free variables to 1 (if there are any free variables) and verify that the resulting solution does indeed work in the equation  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{b}_1$ .

$$\text{Setting } x_3 = 1 \text{ and } x_4 = 1 \text{ we find } x_1 = 5 - 3 - 4 = 5 - 7 = -2 \text{ AND } x_2 = -4 + 2 + 3 = -4 + 5 = 1$$

now let's check that the solution  $(-2, 1, 1, 1)$  "works":

$$-2 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} + 1 \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 6 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 + 2 + 5 + 6 \\ -8 + 4 + 4 + 4 \\ -10 + 7 + 1 - 1 \end{bmatrix} = \begin{bmatrix} -6 + 13 \\ -8 + 12 \\ -11 + 8 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \text{ as expected.}$$

2. In general, an expression such as  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p$  is called a what?

A linear combination of the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ .

3. Let  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  and  $\mathbf{v}_4$  be as in question 1. Let  $\mathbf{b}_2 = \begin{bmatrix} 8 \\ 4 \\ -2 \end{bmatrix}$ . Is  $\mathbf{b}_2$  in  $\text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\})$ ? Fully explain your

answer, and again show any augmented matrices and their RREF's that you use.

Now the corresponding augmented matrix is

$$\left[ \begin{array}{cccc|c} 3 & 2 & 5 & 6 & 8 \\ 4 & 4 & 4 & 4 & 4 \\ 5 & 7 & 1 & -1 & -2 \end{array} \right] \text{ which has RREF } \left[ \begin{array}{cccc|c} 1 & 0 & 3 & 4 & 0 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

The last row of this RREF represents the equation  $0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$ , which

has NO solutions for any choice of  $x_1, x_2, x_3, x_4$ . Thus the equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{b}_2 \text{ also has no solutions, and so } \underline{NO}, \vec{b}_2 \text{ is } \underline{NOT} \text{ in}$$

the span of the vectors  $\vec{v}_1, \dots, \vec{v}_4$ .